STABILITY ANALYSIS OF CRITICAL NONLINEAR SYSTEMS USING NORMAL FORM TECHNIQUES

By

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To My Wife Wei Xiong and My Son Pongpong
&
To My Parents Jingzhong and Baiyin

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STABILITY ANALYSIS OF CRITICAL NONLINEAR SYSTEMS USING NORMAL FORM TECHNIQUES

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Nonlinear dynamical models arise from both theoretical research and real applications, including robotics, aircraft control, adaptive control, chemical process, numerical algorithm, and power systems. The most important analysis for nonlinear systems is the determination of stability. Only critical nonlinear systems need further investigation, and up to now, stability analysis has been predominantly performed using center manifold and bifurcation theories in combination with Lyapunov techniques. When a dynamic system is locally asymptotically stable (LAS in short), a Lyapunov function must exist. Many system properties, such as stability region, convergence rate, and robustness, are associated with and can be estimated using Lyapunov functions. The work on the determination of the local

stability of critical nonlinear systems, or the construction of Lyapunov functions when they are LAS, is far from complete.

In this dissertation, critical nonlinear systems are studied using normal form theory. Once a critical nonlinear system is transformed into a normal form, the dynamics of the critical states are decoupled locally from those of the stable states. Similar to results derived using center manifold theory, the stability of the entire nonlinear system is determined by that of critical dynamics. The relationship between normal form and center manifold theories is then explored. If the critical dynamics is LAS, and a Lyapunov function is available, a Lyapunov function for the entire system can be easily constructed.

The convergence is not always guaranteed for normal forms. Construction of Lyapunov functions for purely critical nonlinear systems is only known in a few special cases. Therefore, critical nonlinear systems in two special cases are studied intensively using approximate normal forms. The explicit stability criteria, as well as explicit algorithms for construction of Lyapunov functions, are developed. Methods for simplification of Lyapunov functions are also proposed. The construction of Lyapunov functions requires deriving (approximate) normal forms and the associated transformations, and is very computationally intensive. The issue of reducing the complexity of computing normal forms and the associated Lyapunov functions is also studied. Critical nonlinear systems with multiple critical eigenvalues are examined, and some sufficient conditions for determining stability and constructing Lyapunov functions are obtained.

This dissertation deals with continuous-time systems. All of the results are applicable to discrete-time systems with slight modifications.

CHAPTER 1
INTRODUCTION

Nonlinear dynamical models arise from both theoretical research and real applications, including robotics, aircraft control, adaptive control, chemical processes, numerical algorithms, and power systems. The most important analysis for nonlinear dynamical systems is the determination of stability. Linear system theory has be well-developed in last decades. To a certain extent, the local stability of a nonlinear system can be completely characterized by its linearization at the equilibrium point of interest. The local stability of a nonlinear system cannot be determined in the case that the linearization possesses at least one critical eigenvalue while all other eigenvalues are stable. This dissertation will study nonlinear dynamical systems whose linearizations exhibit such a property. The local stabilizability of nonlinear control systems possessing linearly uncontrollable critical eigenvalues will also be discussed. The emphasis will be on the construction of Lyapunov functions for critical nonlinear systems which are known to be locally asymptotically stable (LAS). This chapter serves as an introduction to the dissertation. In Section 1.1, we present much of the background material which will be used throughout this dissertation. Section 1.2 gives a brief literature review. The organization, along with the major contributions, of this dissertation is presented in Section 1.3.

1.1 Problem Formulation

In this section, we present much of the background material that will be used throughout this dissertation.

A map $g : \Omega \subset R^m \rightarrow R^n$ is said to be a map of class C^k (or, simply C^k), denoted as $g \in C^k$, for some nonnegative integer k if its partial derivatives of up to order k with respect to its variables exist and are continuous in Ω. If we can always choose $k = +\infty$, then we say that g is a map of class C^∞ (or, simply C^∞). Furthermore, g is said to be analytic (denoted as C^ω) if it is C^∞ and for each point $x_0 \in \Omega$ Taylor series expansion of g at x_0 converges to $g(x)$ for all $x \in \Omega$.

The goal of this dissertation is to study the local asymptotic stability (LASY in short) of nonlinear dynamical systems of form

$$\dot{x} = f(x), \quad x \in R^n \tag{1.1}$$

where $f : \Omega \rightarrow R^n$ is a sufficiently differentiable (or C^k with k sufficiently large) map from a domain $\Omega \subset R^n$ into R^n.

An *equilibrium solution* (or *equilibrium point*, or simply *equilibrium*) of (1.1) is a point $x_0 \in \Omega$ such that

$$f(x_0) = 0,$$

i.e., a solution which does not change in time.

For convenience, we state all the definitions and theorems for the case when the equilibrium point is at the origin of R^n; i.e., $x_0 = 0$. There is no loss of generality in doing this since any equilibrium point can be shifted to the origin via a change of variables [37]. Occasionally, the term "zero solution" is used, instead, to signify that the origin is the equilibrium point of interest.

The linearization of (1.1) is defined as

$$\dot{x} = Ax, \tag{1.2}$$

where $A = Df(0)$ is the first order derivative, or the Jacobian matrix of $f(x) : Df(x) = \partial f(x)/\partial x$, evaluated at the origin. The definition of eigenvalues or eigenvectors is well-known. The spectrum of an $n \times n$ matrix A is the set of eigenvalues of A, and denoted as $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ where λ_i needs not to be different from λ_j. Assume λ is an eigenvalue of the $n \times n$ matrix A. The eigenvalue λ is said to be *hyperbolic* if its real part is nonzero (denoted as $Re(\lambda) \neq 0$), *stable* if its real part is negative (denoted as $Re(\lambda) < 0$), *unstable* if its real part is positive (denoted as $Re(\lambda) > 0$), and *critical* if its real part is zero (denoted as $Re(\lambda) = 0$). A nonlinear system of form of (1.1) is said to be *critical* if its linearization only possesses both stable and critical eigenvalues, and *purely critical* if its linearization only possesses critical eigenvalues. The equilibrium point $x_0 = 0$ is said to be *hyperbolic* if none of the eigenvalues of $A = Df(0)$ have zero real part. Similarly, a nonlinear control system

$$\dot{x} = f(x, u)$$

is said to be critical if the linearization about $x = 0$; $u = 0$ has at least one critical eigenvalue that is not controllable while all other eigenvalues are stabilizable.

The term "stability" has different definitions in different areas. In this dissertation, we study the stability of nonlinear systems in the sense of Lyapunov. Let $x(t)$ denote the solution (trajectory) of the system (1.1) which at the initial time $t_0 = 0$ passes through the initial point $x_0 = x(0)$.

Definition 1 ([37], [28])The equilibrium point $x = 0$ of (1.1) is

- *stable (in the sense of Lyapunov), if for each* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that*

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon, \forall t \geq 0.$$

- *unstable (in the sense of Lyapunov), if it is not stable.*

- *asymptotically stable (in the sense of Lyapunov), if it is stable and* δ *can be chosen such that*

$$||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0.$$

◊◊◊

This definition is mathematically very tidy. However, this definition does not provide us with a method for determining whether or not a given equilibrium point is stable. The method of Lyapunov can often be used to determine the stability of equilibrium points.

Theorem 1 *([37], [28])Consider the nonlinear system (1.1). Let* $x = 0$ *be an equilibrium point of (1.1) and let* $V : D \subset \Omega \to R$ *be a* C^1 *function defined on some neighborhood* D *of* $x = 0$ *such that*

$$\begin{aligned} V(0) &= 0 \quad \text{and} \quad V(x) > 0 \ \ \forall x \neq 0 \\ \dot{V}(x) &\leq 0 \quad \text{in} \quad D - \{0\}. \end{aligned}$$

Then the equilibrium point $x = 0$ *is locally stable. Moreover, if*

$$\dot{V}(x) < 0 \quad \text{in} \quad D - \{0\}$$

then the equilibrium point $x = 0$ *is locally asymptotically stable (LAS in short).* ◊◊◊

We refer to V as a *Lyapunov function.* When the origin $x = 0$ is LAS, we are often interested in determining how far from the origin can the trajectory $x(t)$ be and still converge to the origin as t approaches ∞. This gives rise to the definition of the *region of attraction* (stability region, region of asymptotic stability, domain of attraction, or basin). The region of attraction is defined as the set of all points x_0 such that $\lim_{t\to\infty} x(t) = 0$ if $x(0) = x_0$. Finding the exact region of attraction analytically might be difficult, or even impossible. However, Lyapunov functions can be used to estimate the region of attraction. A nonlinear system is said to be globally asymptotically stable (GAS in short) if and only if its region of attraction is the entire space. The following theorem can be used to determine whether or not a nonlinear system is GAS.

Theorem 2 (Barbasin-Krasovskii Theorem) ([37], [28]) Let the nonlinear system (1.1) is defined over the entire space R^n. Let $V : R^n \to R$ be a C^1 function such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \ \ \forall x \neq 0$$

$$||x|| \to \infty \Longrightarrow V(x) \to \infty$$

$$\dot{V}(0) = 0 \quad \text{and} \quad \dot{V}(x) < 0 \ \ \forall x \neq 0$$

then the equilibrium point $x = 0$ is GAS. ◇◇◇

Remark 1 As mentioned before, equilibrium solutions are solutions which do not change in time. A necessary condition for $x = 0$ to be LAS in Ω (or GAS in R^n) is that $x = 0$ is the only equilibrium point in Ω (or in R^n).

Example 1 (Lyapunov function)

Consider the following system:

$$\begin{aligned} \dot{x} &= y - x^3 \\ \dot{y} &= -x - x^2 y - y^3. \end{aligned}$$

It is easy to verify that this system has only one equilibrium point at $(x, y) = (0, 0)$. Let $V(x, y) = \frac{1}{2}(x^2 + y^2)$. Clearly $V(0, 0) = 0$, $V(x, y) \to \infty$ as $x^2 + y^2 \to \infty$, and $V(x, y) > 0$ $\forall (x, y) \neq 0$. Then

$$\dot{V} = -x^2 y^2 - x^4 - y^4 < 0 \quad \forall (x, y) \neq 0.$$

Therefore the origin is globally asymptotically stable. $\diamond\diamond\diamond$

The linear system theory has been well developed over last decades. To a certain extent, the stability and the instability of a nonlinear system can be completely characterized by its linearization.

Theorem 3 [30]Consider the system (1.1). Let $A = Df(0)$. If all of the eigenvalues of A have negative real parts, then the origin is an LAS equilibrium point. If A possesses any eigenvalue with positive real part, then the origin is unstable. If A only possesses both stable and critical eigenvalues, the stability of (1.1) cannot be deduced from the linearization.

For later purposes, we need the following two theorems which give criteria for determining local stability of nonlinear systems in two special cases.

Theorem 4 [49] A one dimensional analytic system of form

$$\dot{x} = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is LAS if and only if $a_k < 0$ and k is odd where $k = \min\{i \mid a_i \neq 0\}$. i.e., the first nonzero number in the sequence $\{a_1, a_2, a_3, \cdots,\}$ is negative and the subscript is odd. $\Diamond\Diamond\Diamond$

This theorem can be easily proved using the Lyapunov function $V(x) = \frac{1}{2}x^2$.

Now consider a two-dimensional analytic system of form with $\omega \in R^+$

$$\begin{aligned}\dot{x} &= -\omega y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ &\quad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + O(|x,y|^4) \qquad (1.3)\\ \dot{y} &= \omega x + b_{20}x^2 + b_{11}xy + b_{02}y^2 \\ &\quad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + O(|x,y|^4). \qquad (1.4)\end{aligned}$$

After a proper change of coordinates $(x, y) \to (\xi, \eta)$, we obtain

$$\begin{aligned}\dot{\xi} &= -\omega\eta + \sum_{i=1}^{N}(\xi^2+\eta^2)^i(m_{2i+1}\xi - n_{2i+1}\eta) + O(|\xi,\eta|^{2N+2}) \qquad (1.5)\\ \dot{\eta} &= \omega\xi + \sum_{i=1}^{N}(\xi^2+\eta^2)^i(m_{2i+1}\eta + n_{2i+1}\xi) + O(|\xi,\eta|^{2N+2}). \qquad (1.6)\end{aligned}$$

Theorem 5 ([6], [27])The system (1.5), (1.6)(or equivalently (1.3), (1.4)) is LAS if and only if $m_{2k+1} < 0$ where $k = \min\{i \mid m_{2i+1} \neq 0\}$, i.e., the first nonzero number in the sequence $\{m_3, m_5, m_7, \cdots,\}$ is negative. $\Diamond\Diamond\Diamond$

This theorem can be easily proved using the Lyapunov function $V(\xi, \eta) = \frac{1}{2}(\xi^2+\eta^2)$ for the system (1.5), (1.6). As we will see later, the system (1.5) and (1.6) is one of the normal forms of (1.3) and (1.4). For a given system, the computation of m_{2i+1} is quite involved but systematic. For example,

$$m_3 = \frac{1}{16}(a_{30} + a_{12} + b_{21} + b_{30})$$

$$+\frac{1}{16\omega}\left[a_{11}(a_{20}+a_{02})-b_{11}(b_{20}+b_{02})-a_{20}b_{20}+a_{02}b_{02}\right].$$

1.2 Brief Literature Review

Stability (or stabilizability) of critical nonlinear (control) systems is a classical problem in both theoretical research ([6], [38], [53], [16], [15]) and variety of applications ([41], [43], [40], [46], [52], [45]). Many different approaches are used to study nonlinear stability (or stabilizability) analysis. In particular, center manifold theory is used in [3], [11], [6], [39], and [7]. In [1], [2], [25], [32], [33], [34], [35], [41], [43], and [52], bifurcation theory is employed. Even though center manifold and bifurcation theories are successfully used in stability (or stabilizability) analysis; these techniques are found (up to the date this dissertation is finished, and to author's best knowledge) not very helpful for the construction of Lyapunov functions when a critical nonlinear system is LAS.

In this dissertation, we propose to study critical nonlinear systems using Poincaré normal form theory. The normal form method is a technique of change of coordinates, and is capable of bringing out the essential features of nonlinear systems. The technique of using change of coordinates in nonlinear system analysis is not new. Lyapunov already used technique of change of coordinates in stability analysis of nonlinear systems in his landmark Ph.D. dissertation [42] in 1892. Different normal form techniques are discussed in much of literature, such as [4], [5], [8], [9], [14], [19], [20], [17], [18], [21], [22], [36], [31], [50]. Roughly speaking, the linearization of affine nonlinear systems [29], [47] also belongs to the area of normal form techniques. In [6], and [38], the Poincaré normal form is employed for the stability analysis of critical nonlinear systems. Behtash and Sastry [7] examined nonlinear (control) systems using both normal form and center manifold theories. However, the power of normal form techniques has not yet been fully recognized and exploited. The

search to discover other potential merits of normal form techniques in study of nonlinear (control) systems motivates much of the work presented in this dissertation.

When a dynamical system is LAS, the existence of Lyapunov functions is guaranteed. For linear systems or nonlinear systems whose linearization possesses no critical eigenvalues, the algorithm for the construction of Lyapunov functions is well established. Many system properties such as stability region, convergence rate, and robustness can be estimated using Lyapunov functions (see [26], [12], [13] for example). Unfortunately, there is no general procedure for the construction of Lyapunov functions for critical nonlinear systems. In [44], Mees and Chua constructed a Lyapunov function for a simple 2-dimensional nonlinear system with a pair of purely imaginary eigenvalues. The method they used cannot be generalized. Fu and Abed [24] proposed a procedure for the construction of Lyapunov functions for nonlinear systems in two special cases. The procedure is based on a lemma about the local definiteness of a class of bivariate functions, and is only applicable when the local stability of a nonlinear system can be determined using only the first through the third order terms without necessity of the consideration of the higher order terms. However, with the help of Poincaré normal form techniques, the algorithm proposed in this dissertation is able to construct Lyapunov functions for nonlinear systems in two special cases no matter how many terms in the dynamics are necessary to determine the local stability. Other nonlinear systems with more than one (or one pair of) critical eigenvalue(s) is also studied using normal form techniques.

1.3 Outline of the Dissertation

Aside from the first and last Chapters, this dissertation can be divided into two parts. The mathematical preliminaries are introduced in Chapters 2 and 3, while most of results from this research are presented in Chapters 4 through 8.

In Chapter 2, center manifold theory is introduced. Center manifold theory is a prevailing model reduction method in analysis of critical nonlinear systems. We include the definition of center manifold and three fundamental theorems of the center manifold theory in this chapter. Several examples are given to illustrate use of the center manifold theory.

Chapter 3 presents the introduction to normal form theory. In essence, normal form theory is a change of coordinates technique. We first illustrate the concept and structure of normal forms using near identity of changes of coordinates. An efficient algorithm for the computation of normal forms follows. By introducing and studying a linear operator in the space of homogeneous maps with a properly defined inner product, the structure of normal forms can be predicted. The space of homogeneous maps is a linear one. It is also possible to study normal forms in other linear spaces once a correspondence is established between the two linear spaces.

In Chapter 4, we investigate the relationship between center manifold and normal form theories. Center manifolds for a given critical nonlinear system need not be unique. However, it is shown that all center manifolds are equivalent in some sense. When normal form theory is applied to a critical nonlinear system, the system in normal forms exhibits a decoupled-like structure. The stability of the entire system is determined by that of the critical dynamics. This conclusion is identical to the results derived using center manifold theory. The relationship between center manifold and normal form theories is then explored.

The merits of normal forms are exploited further in Chapter 5. We develop procedures for construction of Lyapunov functions for critical nonlinear systems when they are locally asymptotically stable. First, a general procedure is provided. Since the critical dynamics are independent of stable dynamics for systems in normal forms (when they exist), Lyapunov functions for the entire system can be derived by just summing the individual Lyapunov functions corresponding to critical and stable dynamics, respectively. However, the normal form procedure is not always convergent, and Lyapunov functions for purely critical systems are not always available. Nevertheless, we are able to develop an algorithm for the construction of Lyapunov functions for critical nonlinear systems in two specific cases using approximate normal forms.

Compared to the Lyapunov functions constructed by other researchers for some simple examples, the Lyapunov functions constructed in Chapter 5 are quite complicated in some sense. In Chapter 6, an algorithm for the simplification of Lyapunov functions is proposed. Several examples are given to highlight the simplicity of these refined Lyapunov functions.

An alternative algorithm (compared to the one given in Chapter 5) is proposed in Chapter 7. To use this algorithm, a modified (partial) approximate normal form is sufficient for the construction of Lyapunov functions. This will significantly reduce the complexity of computing normal forms and the associated transformations. The simplification of the associated Lyapunov functions is also addressed.

Chapter 8 includes several independent ideas. We apply normal form techniques to a broader class of critical nonlinear systems than those considered in previous chapters. Under certain hypotheses, the stability of such systems can be easily determined and the associated Lyapunov functions can be easily constructed. The bulk of the research presented in this dissertation only treats systems in the continuous-time domain. Actually, normal

form techniques are applicable to discrete-time systems using a slight modification. An brief introduction to use of normal form theory in the discrete-time domain concludes Chapter 8.

Finally, in Chapter 9, we summarize the dissertation and give some concluding remarks.

CHAPTER 2
INTRODUCTION TO CENTER MANIFOLD THEORY

The center manifold theorem in finite dimensions can be traced to the work of Pliss [48]. Other valuable references include [11], [53], [27], and [15].

Center manifold theory provides a model reduction technique for analyzing critical nonlinear dynamic systems. As we have mentioned before, a necessary condition for a nonlinear system to be LAS is that the linearization possesses no eigenvalues with positive real parts. A nonlinear dynamic system is LAS if all eigenvalues of the linearization have negative real parts. The only case when the local stability of a nonlinear system cannot be deduced from the linearization is when the linearization possesses both stable and critical eigenvalues. In this situation, nonlinear terms must be taken into consideration for determining the stability of a nonlinear dynamical system. Center manifold theory allows us to deduce the stability of a nonlinear system by studying a purely critical nonlinear system which in general has a reduced dimension.

The definition of the center manifold is given below first.

Definition 2 [53, 11] (Invariant Manifold) A set $S \in R^n$ is said to be a local invariant manifold for the dynamical system

$$\dot{x} = f(x), \quad x \in R^n \tag{2.1}$$

if for $x_0 \in S$, the solution $x(t)$ of (2.1) with $x(0) = x_0$ is in S for $|t| < T$ where $T > 0$. If we can always choose $T = +\infty$, then we say that S is an invariant manifold. ◇◇◇

Definition 3 [53, 11] (Center Manifold) An invariant manifold S is called a center manifold for (2.2)-(2.3) if it can be locally represented as follows

$$S = \{(x_c, x_s) \in R^{n_c} \times R^{n_s} \mid x_s = h(x_c); \quad |x| < \varepsilon; \quad h(0) = 0; \quad Dh(0) = 0\}$$

for some $\varepsilon > 0$. ◇◇◇

Center manifold theory has several applications to dynamical systems. In this research, center manifold theory is employed for determining the stability of dynamical systems of form

$$\dot{x}_c = A^c x_c + f_c(x_c, x_s) \tag{2.2}$$

$$\dot{x}_s = A^s x_s + f_s(x_c, x_s), \tag{2.3}$$

where $x_c \in R^{n_c}$, $x_s \in R^{n_s}$, A^c and A^s are constant matrices such that all of the eigenvalues of A^c have zero real parts while all of the eigenvalues of A^s have negative real parts. The vector valued functions f_c, f_s are at least C^k $(k \geq 2)$ with $f_c(0,0) = 0$, $f_s(0,0) = 0$, $Df_c(0,0) = 0$, $Df_s(0,0) = 0$.

The three fundamental theorems of the center manifold theory are summarized as follows:

Theorem 6 [11](Existence of Center Manifolds) There exists a C^k center manifold $y = h(x)$ for (2.2)-(2.3). The dynamics of (2.2)-(2.3) restricted to the center manifold is, for ξ

sufficiently small, given by the following n_c-dimensional system

$$\dot{\xi} = A^c \xi + f_c(\xi,\ h(\xi)),\ \xi \in R^{n_c}. \tag{2.4}$$

◇◇◇

Theorem 7 [11](Reduction Principle) (I) Suppose the zero solution of (2.4) is stable (or asymptotically stable, or unstable). Then the zero solution of (2.2)-(2.3) is also stable (or asymptotically stable, or unstable). (II) Suppose the zero solution of (2.4) is stable. Let $(x_c(t),\ x_s(t))$ be a solution of (2.2)-(2.3) with $(x_c(0),\ x_s(0))$ sufficiently small. Then there is a solution of (2.4) such that as $t \to +\infty$

$$\begin{aligned} x_c(t) &= \xi(t) + O(e^{-\gamma t}) \\ x_s(t) &= h(\xi(t)) + O(e^{-\gamma t}) \end{aligned}$$

where $\gamma > 0$ is a constant. ◇◇◇

Substituting the center manifold $x_s(t) = h(x_c(t))$ into (2.3), we obtain

$$Dh(x_c)[A^c x_c + f_c(x_c, h(x_c))] = A^s h(x_c) + f_s(x_c, h(x_c)). \tag{2.5}$$

The center manifold $y = h(x)$ is obtained by solving equation (2.5) together with the conditions $h(0) = 0$, $Dh(0) = 0$. A closed form solution to the PDE (2.5) is often not easily found. The next theorem, however, shows that the center manifold can be approximated, in principle, to any degree of accuracy.

Theorem 8 [11](Approximation of the Center Manifold) Let $\phi : D \subset R^{n_c} \to R^{n_s}$ be a C^1 mapping with $\phi(0) = 0$ and $D\phi(0) = 0$. Suppose that as $x_c \to 0$

$$(M\phi)(x_c) \triangleq D\phi(x_c)[A^c x_c + f_c(x_c, \phi(x_c))] - A^s\phi(x_c) - f_s(x_c, \phi(x_c)) = O(||x_c||^q)$$

for some $q > 1$. Then as $x_c \to 0$, $||h(x_c) - \phi(x_c)|| = O(||x_c||^q)$. ◇◇◇

Remark 2 [11] Center manifolds need not to be unique. However, there are some points which must always lie on the any center manifold. For example, any center manifold must include the equilibrium point. ◇◇◇

Remark 3 [11] If f_c and f_s are C^k, $(2 \leq k < \infty)$, then h is also C^k. If f_c and f_s are analytic, then in general (2.2)-(2.3) does not necessarily have an analytic center manifold, not even a C^∞ one. ◇◇◇

A few examples are provided to illustrate the use of center manifold theory.

Example 2 (Derivation of the center manifold)

Consider the system

$$\dot{x}_c = x_c x_s + a x_c^3 + b x_c x_s^2 \tag{2.6}$$

$$\dot{x}_s = -x_s + c x_c^2 + d x_c^2 x_s + e x_c^3, \tag{2.7}$$

where a, b, c, d, and e are real constants.

By Theorem 6, the system (2.6)-(2.7) has a center manifold $x_s = h(x_c)$. Assuming $h(x_c) = h_2 x_c^2 + h_3 x_c^3 + h_4 x_c^4 + h_5 x_c^5 + \cdots$, we obtain

$$(Mh)(x_c) = (-c + h_2)\, x_c^2 + (h_3 - e)\, x_c^3$$

$$+ (2\,h_2\,(h_2 + a) + h_4 - dh_2)\,x_c^4$$

$$+ (h_5 - dh_3 + 2\,h_2 h_3 + 3\,h_3\,(h_2 + a))\,x_c^5 + O\left(\|x_c\|^6\right).$$

Hence, $(Mh)(x_c) = O\left(\|x_c\|^6\right)$ if we take

$$h(x_c) = cx_c^2 + ex_c^3 + (dc - 2c(c+a))x_c^4 + (de - 2ce - 3e(c+a))x_c^5 + \cdots.$$

By Theorem 7, the dynamics which determines the stability of the zero solution of the system (2.6)-(2.7) is

$$\dot{\xi} = (c+a)\,\xi^3 + e\xi^4 + \left(dc - 2\,c\,(c+a) + bc^2\right)\xi^5 + O\left(\|\xi\|^6\right).$$

◇◇◇

Example 3 [11] (Non-uniqueness of the center manifold)

Consider the system

$$\dot{x}_c = -x_c^3 \tag{2.8}$$

$$\dot{x}_s = -x_s. \tag{2.9}$$

It is easy to verify that for any real number c_1, c_2,

$$x_s = h\,(x_c, c_1, c_2) = \begin{cases} c_1 \exp(-\frac{1}{2}x_c^{-2}), & x_c > 0 \\ 0 & x_c = 0 \\ c_2 \exp(-\frac{1}{2}x_c^{-2}), & x_c < 0 \end{cases}$$

satisfies (2.5), hence is a center manifold for the system (2.8)-(2.9). ◇◇◇

Example 4 [51] (Non-analyticity of the center manifold)

Consider the system

$$\dot{x}_c = -x_c^3 \tag{2.10}$$

$$\dot{x}_s = -x_s + x_c^2. \tag{2.11}$$

Suppose that the system (2.10)-(2.11) has a center manifold which is analytic at the origin. Then, by Taylor's Theorem,

$$h(x_c) = \sum_{n=2}^{\infty} h_n x_c^n \tag{2.12}$$

in some neighborhood of the origin. The coefficients $\{h_n\}$ are determined by substituting (2.12) into (2.5). We have,

$$h_2 = 1$$

$$h_{2n+1} = 0, \quad \text{for} \quad n = 1,\ 2,\ 3, \cdots$$

$$h_{n+2} = nh_n, \quad \text{for} \quad n = 2,\ 4,\ 6,\ \cdots .$$

A series of the form (2.12) with these coefficients is not convergent. Therefore the system (2.10)-(2.11) cannot have an analytic center manifold. ◇◇◇

Example 5 (Nonlinear systems depending on parameters)

Consider the system

$$\dot{x} = \varepsilon x - x^3 + xy \tag{2.13}$$

$$\dot{y} = -y + y^2 - x^2 \tag{2.14}$$

where ε is a real constant parameter.

It is evident that the origin (0,0) is an isolated equilibrium point for the system (2.13)-(2.14). The eigenvalues of the Jacobian matrix corresponding to this equilibrium point are -1 and ε. Center manifold techniques cannot be applied to (2.13)-(2.14) directly since the sign of ε is unknown. However, we can re-write (2.13)-(2.14) in an equivalent form as follows

$$\dot{x} = \varepsilon x - x^3 + xy \tag{2.15}$$

$$\dot{y} = -y + y^2 - x^2 \tag{2.16}$$

$$\dot{\varepsilon} = 0. \tag{2.17}$$

When considered as an equation on R^3 the εx term in (2.15) is nonlinear. Thus $\{-1, 0, 0\}$ is the set of eigenvalues for the Jacobian matrix corresponding (2.15)-(2.17) at the origin (0,0,0). Now center manifold theory can be applied. Assume a center manifold for the system (2.15)-(2.17) is

$$y = h(x, \varepsilon) = a_{20}x^2 + a_{11}x\varepsilon + a_{02}\varepsilon^2 + \cdots,$$

we obtain

$$(Mh)(x, \varepsilon) = (1 + a_{20})x^2 + a_{11}x\varepsilon + a_{02}\varepsilon^2 + O(|x, \varepsilon|^3).$$

Thus $y = h(x, \varepsilon) = -x^2 + \cdots$ is a 2nd order approximation to center manifolds. Since for any value of ε, $(0, 0, \varepsilon)$ is always an equilibrium point for the system (2.15)-(2.17) (see

Remark 2), we conclude that $h(0,\varepsilon) = 0$. The dynamics along the center manifold is

$$\begin{aligned} \dot{\xi} &= \varepsilon\xi - 2\xi^3 + O(|\xi| \cdot |\xi, \varepsilon|^3) \\ \dot{\varepsilon} &= 0. \end{aligned}$$

It is easy to conclude now that the system (2.15)-(2.17) is LAS if $\varepsilon \leq 0$, and unstable if $\varepsilon > 0$. ◇◇◇

As we remarked before, center manifolds need not to be unique for a given nonlinear system. However, it will be shown in Chapter 4 that all of the center manifolds for a given nonlinear system are equivalent in some sense.

CHAPTER 3
INTRODUCTION TO NORMAL FORM THEORY

This chapter is concerned with normal form theory. The method of normal forms can be traced to the Ph.D thesis of Poincaré, is further refined in [50]. Other good references include [4], [17], [17], [15], and [10]. In Section 3.1, the concept as well as structure of normal forms are introduced. Derivation of normal forms is very computationally intensive. Efficient algorithms are presented in Section 3.2. In Section 3.3, we give a rigorous definition of the normal form, and provide some observations about the linear operator L_A^k. Several examples are provided in this chapter as an illustration of the use of normal form theory. A brief summary of this chapter is given in Section 3.4.

3.1 Introduction to Normal Forms

The concept as well as structure of normal forms can be better illustrated by applying a sequence of *near identity changes of coordinates* to a nonlinear system.

To simplify the description of normal forms for nonlinear dynamical systems, it is first necessary to introduce some basic notation about homogeneous polynomial maps of a given degree. Other related notation will be defined sequentially as we go along. All of this notation is defined over the fields of real numbers, but is also valid over the field of complex numbers.

For a positive integer $k \geq 0$, a *monomial of degree k* over R^n is an expression

$$x^\alpha \triangleq x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad \text{with} \quad |\alpha| \triangleq \sum_{i=1}^{n} \alpha_i = k$$

where $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]$ (with integer $\alpha_i \geq 0$) is a multi-index.

A *homogeneous polynomial function of degree* k, $f : R^n \to R$ is defined as the linear combination of all monomials of degree k over R^n, i.e.,

$$f = \sum_{|\alpha|=k} b_\alpha x^\alpha$$

where $b_\alpha \in R$. A vector-valued polynomial function (map) $f : R^n \to R^m$ is said to be homogeneous of degree k if every entry is a homogeneous polynomial function of degree k.

In this dissertation, a homogeneous polynomial map (function) of degree k will be denoted as f^k with the superscript indicating the degree. Notice that zero is a homogeneous polynomial map (function) of any degree.

The space of homogeneous polynomial maps from R^n to R^m of degree k is defined as

$$H(n, m, k) = \{f^k : R^n \to R^m\}.$$

Therefore, $H(n, m, k)$ is a linear space with $\dim H(n, m, k) = m\binom{n+k-1}{k} = m\frac{(n+k-1)!}{k!(n-1)!}$. An inner product $\langle \cdot, \cdot \rangle$ on the space $H(n, m, k)$ is introduced in [4].

Let A be an $n \times n$ real constant matrix. For each $k \geq 2$, the linear operator $L_A^k : H(n, n, k) \to H(n, n, k)$ is defined by

$$(L_A^k P^k)(y) = DP^k(y)Ay - AP^k(y) = [Ay, P^k(y)].$$

This operator will be used in constructing normal forms.

Let $R(n,n,k)$ denote the range of L_A^k, and $C(n,n,k)$ any complementary (not necessarily the orthogonal) subspace of $R(n,n,k)$ in $H(n,n,k)$. This sets up the decomposition

$$H(n,n,k) = R(n,n,k) \oplus C(n,n,k). \tag{3.1}$$

Now we are in the position to introduce normal form theory. Without loss of generality, consider an analytic nonlinear dynamical system expanded as a series of homogeneous polynomial maps as follows:

$$\dot{x} = f(x) = f^1(x) + f^2(x) + \cdots + f^{k-1}(x) + f^k(x) + f^{k+1}(x) + \cdots, \qquad x \in R^n, \tag{3.2}$$

where $f^i(\cdot) \in H(n,n,i)$. Here $f^1(x) = Ax$ with $A = Df(0)$.

For $P^k(\cdot) \in H(n,n,k)$, consider now the effect of a *near identity change of coordinates* of the form

$$x = y + P^k(y). \tag{3.3}$$

Differentiating both sides of (3.3) with respect to time and using (3.2) yields

$$(I + DP^k(y))\,\dot{y} = f(y + P^k(y)). \tag{3.4}$$

Note that $(I + DP^k(y))$ is always invertible in a sufficiently small neighborhood of the origin, and

$$(I + DP^k(y))^{-1} = I - DP^k(y) + O(|y|^{2k-2}), \tag{3.5}$$

where $O(|y|^{2k-2})$ denotes combination of terms of degree $(2k-2)$ or higher.

Equations (3.4), (3.5) and (3.2) yields the dynamics:

$$\dot{y}= g(y) = g_k^1(y) + g_k^2(y) + \cdots + g_k^{k-1}(y) + \overbrace{f^k(y) - [Ay, P^k(y)]} + g_k^{k+1}(y) + \cdots, \qquad (3.6)$$

where $g_k^i(\cdot) = f^i(\cdot)$ for $i = 1, 2, \cdots, k-1$. In general, for $i \geq k+1$, $g_k^i(\cdot) \neq f^i(\cdot)$ in the equation (3.2). Here, the subscript 'k' corresponds to the fact that the transformation $P^k \in H(n, n, k)$. The term of degree k in (3.6) is given by

$$g_k^k(y) = f^k(y) - [Ay, P^k(y)] = f^k(y) - (L_A^k P^k)(y). \qquad (3.7)$$

This suggests introducing the linear operator L_A^k: $H(n, n, k) \to H(n, n, k)$ for each $k \geq 2$.

If $f^k \in R(n, n, k)$, we can choose an appropriate $P^k(\cdot)$ such that $(L_A^k P^k)(y) = f^k(y)$, and consequently completely cancel out all terms of degree k in (3.6). In the more general cases where $f^k \notin R(n, n, k)$, we can always pick up an appropriate $P^k(\cdot)$ such that $f^k(y) - [Ay, P^k(y)] \in C(n, n, k)$. The terms of degree k which cannot be removed should be considered in some sense as the **essential** part for the dynamical behavior of (3.2) near the origin.

In order to implement the normal form transformation, apply a sequence of changes of coordinates (3.6) to the system (3.2) iteratively, starting with $k = 2$. At each step, an appropriate P^k is chosen such that $g_{k-1}^k(y) - [Ay, P^k(y)] \in C(n, n, k)$. After a finite number of iterations, the system (3.2) is said to be put into an approximate normal form. Clearly, the process of "normalization" is intrinsically local in the sense that the change of coordinates is only valid in some neighborhood of the origin. The following normal form theorem is obtained from the above discussion.

Theorem 9 (Normal Form Theorem)[4], [6] For the system (3.2), given the sequence of decompositions (3.1) and any finite positive integer k, there exists a series of changes of coordinates, $x_{i-1} = x_i + P^i(x_i)$, $x_1 = x$, $x_k = y$, $i = 2, 3, \cdots, k$, $P^i \in H(n, n, i)$, and a neighborhood Ω of the origin, such that for $y \in \Omega$, the system (3.2) is transformed to a system of the form

$$\dot{y} = g^1(y) + g^2(y) + \cdots + g^k(y) + O(||y||^{k+1}), \tag{3.8}$$

where $g^1(y) = Ay$ and $g^i(y) \in C(n, n, i)$ for $k = 2, 3, \cdots, k$. ◇◇◇

Remark 4 Clearly the structure, as well as actual expression, of the normal form (3.8) of (3.2) depends on A. Even if A is fixed, the normal form is still not unique, but depends on the choice of complementary subspaces $C(n, n, i)$, as well as other factors. A more rigorous definition of the normal form will be given later. ◇◇◇

Remark 5 As long as the number of the iterations of the changes of coordinates is finite, the normal form (3.8) of (3.2) must be convergent in some neighborhood Ω of the origin. If the sequence of changes of coordinates is taken to infinity, the resulting sequence of transformations may not be convergent. In the case that the resulting normal form process is convergent, and $f \in C^\omega$, the resulting full normal form is given by:

$$\dot{y} = g(y) = g^1(y) + g^2(y) + g^3(y) + \cdots \tag{3.9}$$

where $g^1(y) = Ay$ and $g^k \in C(n, n, k)$ for $k = 2, 3, \cdots$. ◇◇◇

Example 6 (Derivation of Normal Forms)

Consider the system

$$\dot{x} = xy + x^3 \tag{3.10}$$

$$\dot{y} = -y - 2x^2, \tag{3.11}$$

where $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. For terms of degree two, we have

$$H(2,2,2) = span\left\{ \begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \right\}.$$

We want to compute $L_A^2(H(2,2,2))$. We do this by applying the linear operator L_A^2 to each basis element on $H(2,2,2)$. These are:

$$L_A^2 \begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -y \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$L_A^2 \begin{pmatrix} xy \\ 0 \end{pmatrix} = \begin{pmatrix} y & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -y \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} xy \\ 0 \end{pmatrix} = -\begin{pmatrix} xy \\ 0 \end{pmatrix}$$

$$L_A^2 \begin{pmatrix} y^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 2y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -y \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y^2 \\ 0 \end{pmatrix} = -2\begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$

$$L_A^2 \begin{pmatrix} 0 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2x & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -y \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$$

$$L_A^2 \begin{pmatrix} 0 \\ xy \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & x \end{pmatrix} \begin{pmatrix} 0 \\ -y \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$L_A^2 \begin{pmatrix} 0 \\ y^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2y \end{pmatrix} \begin{pmatrix} 0 \\ -y \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ y^2 \end{pmatrix} = -\begin{pmatrix} 0 \\ y^2 \end{pmatrix}$$

Therefore $R(2,2,2) = span\left\{\begin{pmatrix} xy \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y^2 \end{pmatrix}\right\}$, $C(2,2,2)$ can be chosen as $C(2,2,2) = span\left\{\begin{pmatrix} x^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xy \end{pmatrix}\right\}$.

Similarly, for terms of degree three, we obtain:

$$H(2,2,3) = span\left\{\begin{pmatrix} x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2y \\ 0 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 \end{pmatrix}\right\},$$

$$R(2,2,3) = span\left\{\begin{pmatrix} x^2y \\ 0 \end{pmatrix}, \begin{pmatrix} xy^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y^3 \end{pmatrix}\right\},$$

and $C(2,2,3)$ may be chosen as

$$C(2,2,3) = span\left\{\begin{pmatrix} x^3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2y \end{pmatrix}\right\}.$$

For $k \geq 4$, $H(2,2,k)$, $R(2,2,k)$, $C(2,2,k)$ can be derived accordingly. Under such decompositions, the normal form up to degree 5 will take the form:

$$\begin{aligned} \dot{x} &= a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \\ \dot{y} &= -y + b_2xy + b_3x^2y + b_4x^3y + b_5x^4y \end{aligned}$$

The actual description of the normal form is not unique. Two possible descriptions of the normal forms up to degree 5 are given as follows:

$$\begin{aligned} \dot{x} &= -x^3 - 4x^5 \\ \dot{y} &= -y + 4xy^2 + 8xy^4 \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= -x^3 - 4x^5 \\ \dot{y} &= -y + 4xy^2 + 20xy^4. \end{aligned}$$

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The fact that $g^k \in C(n,n,k)$ means that g^k is just a linear combination of the basis elements which span $C(n,n,k)$. To determine the structure for g^k, we need to choose the space $C(n,n,k)$ and its basis. The computation shown above is tedious, prone to mistakes, and not systematic. Fortunately, one can easily find $C(n,n,k)$ and the associated basis. Theorem 10 below presents a method for finding $C(n,n,k)$ for a given matrix A. The proof of this theorem is sketched [4].

Theorem 10 *[4]$Ker(L^k_{A^*})$ is an orthogonal complementary subspace to $R(n,n,k)$ in $H(n,n,k)$, i.e.,*

$$H(n,n,k) = R(n,n,k) \oplus Ker(L^k_{A^*}),$$

where $Ker(L^k_{A^*}) \triangleq \left\{\xi^k \mid (L^k_{A^*}\xi^k)(y) = 0\right\}$ *and* $A^* = \overline{A}^T$, *i.e., transpose of conjugate of A.*

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Example 7 *(Use of $Ker(L^k_{A^*})$)*

We use Theorem 10 to find $C(2,2,k)$ for systems with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Let $\xi^k(y) = \begin{pmatrix} \xi_1^k(y_1, y_2) \\ \xi_2^k(y_1, y_2) \end{pmatrix}$. Then

$$
\begin{aligned}
L_{A^*}^k(\xi^k(y)) &= \begin{pmatrix} \partial\xi_1^k/\partial y_1 & \partial\xi_1^k/\partial y_2 \\ \partial\xi_2^k/\partial y_1 & \partial\xi_2^k/\partial y_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1^k(y_1, y_2) \\ \xi_2^k(y_1, y_2) \end{pmatrix} \\
&= \begin{pmatrix} y_1\, \partial\xi_1^k/\partial y_2 \\ y_1\, \partial\xi_2^k/\partial y_2 - \xi_1^k \end{pmatrix}.
\end{aligned}
$$

It is easy to see that the solutions to the equation $L_{A^*}^k(\xi^k(y)) = 0$, i.e.,

$$
\begin{cases} y_1\, \partial\xi_1^k/\partial y_2 = 0 \\ y_1\, \partial\xi_2^k/\partial y_2 - \xi_1^k = 0 \end{cases}
$$

are

$$
\begin{aligned}
\xi_1^k(y_1, y_2) &= a_k\, y_1^k \\
\xi_2^k(y_1, y_2) &= a_k\, y_1^{k-1}\, y_2 + b_k\, y_1^k.
\end{aligned}
$$

Thus $C(2,2,k) = span\left\{ \begin{pmatrix} y_1^k \\ y_1^{k-1}\, y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^k \end{pmatrix} \right\}$. Therefore the structure of the normal form up to degree k is

$$
\begin{aligned}
\dot{y}_1 &= y_2 + a_2 y_1^2 + a_3 y_1^3 + \cdots + a_k\, y_1^k \\
\dot{y}_2 &= (b_2 y_1^2 + a_2 y_1 y_2) + (b_3 y_1^3 + a_2 y_1^2 y_2) + \cdots + (b_k\, y_1^k + a_k\, y_1^{k-1}\, y_2).
\end{aligned}
$$

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In the example 7, it is observed that the dimension of $C(2,2,k)$ is two for any integer $k \geq 2$. However, for each $k \geq 2$, there are three monomials of degree k in normal forms. In many situations, it is desirable to further reduce the number of monomials presented in the dynamics of normal forms. The next theorem tells how one can make such reductions.

Theorem 11 [4]Let $d_k = \dim Ker(L_{A^*}^k)$ *and* $\{v_1, \cdots, v_{d_k}\}$ *be a basis of* $Ker(L_{A^*}^k)$. *If* $\{w_1, \cdots, w_{d_k}\} \subseteq H(n,n,k)$ *satisfies*

$$\langle v_i,\ w_j \rangle = \begin{cases} \delta_i \neq 0 & if\ i = j \\ 0 & otherwise \end{cases} \qquad i,\ j, = 1, 2, \cdots, d_k,$$

then the span$\{w_1, \cdots, w_{d_k}\} \subset H(n,n,k)$ *is also a complementary subspace to* $R(n,n,k)$ *in* $H(n,n,k)$.

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Example 8 (Further Simplification of Normal Forms)

In the previous example, $\{v_1, v_2\} = \left\{ \begin{pmatrix} y_1^k \\ y_1^{k-1} y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^k \end{pmatrix} \right\}$ is a set of basis of $C(2,2,k)$.

Take $\{w_1, w_2\} = \left\{ \begin{pmatrix} 0 \\ y_1^{k-1} y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^k \end{pmatrix} \right\}$. It is easy to verify that for $i, j = 1, 2$, $\langle v_i,\ w_j \rangle \neq 0$ only if $i = j$, using the definition of the inner product introduced in [4]. Therefore, a different normal form up to degree k is given by

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= (b_2 y_1^2 + a_2 y_1 y_2) + (b_3 y_1^3 + a_2 y_1^2 y_2) + \cdots + (b_k\, y_1^k + a_k\, y_1^{k-1}\, y_2)\,. \end{aligned}$$

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Example 9 (Convergence of Normal Forms)

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= \sqrt{2} x_2. \end{aligned}$$

One of the full normal forms is given by

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= \sqrt{2} y_2, \end{aligned}$$

which is obtained using the transformation

$$\begin{aligned} x_1 &= y_1 e^{y_2/\sqrt{2}} \\ x_2 &= y_2. \end{aligned}$$

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Example 10 [10] (Non-convergence of Normal Forms)

In this example it will be shown that the normal form transformation is not always convergent.

The system

$$\begin{aligned} \dot{x}_1 &= x_1^2 \\ \dot{x}_2 &= x_2 - x_1 \end{aligned}$$

can be transformed to

$$\dot{y}_1 = y_1^2$$

$$\dot{y}_2 = y_2$$

using the transformation

$$x_1 = y_1$$

$$x_2 = y_2 + \sum_{k=1}^{\infty} (k-1)!\, y_1^k.$$

Unfortunately, this transformation is not convergent in any small neighborhood of the origin. ◊◊◊

3.2 Computation of Normal Forms

In this Section, we present an efficient algorithm for deriving normal forms and the associated transformations. A general algorithm for obtaining the transformed dynamical system from the original one is introduced first.

3.2.1 A General Algorithm for Coordinate Transformation

Consider a nonlinear dynamical system depending on a parameter as follows,

$$\dot{x} = f(x, a) \tag{3.12}$$

where $f : R^n \times R \to R^n$ and $0 \neq a \in R$.

Let

$$x = u(y, a) \tag{3.13}$$

be a transformation (henceforth called "backward transformation") with $u(y,0) = y$, with its inverse (henceforth called "forward transformation") given by

$$y = v(x,a). \tag{3.14}$$

Clearly, we also have $v(x,0) = x$.

If the transformation (3.13) is applied to the dynamical system (3.12), then the dynamics for y is

$$\dot{y} = g(y,a) \triangleq \frac{\partial v(x,a)}{\partial x} f(x,a)|_{x=u(y,a)}. \tag{3.15}$$

It is desirable to have an efficient algorithms to derive equations (3.15) from (3.12) using (3.13) and/or and (3.14). To simplify the description of the algorithms to be derived, the following partial differential equation (PDE) is introduced:

$$\frac{\partial u}{\partial a} = U(u,a),\ u(y,0) = y. \tag{3.16}$$

Let $u(y,a)$ be the (assumed to be unique) solution to the PDE. The backward transformation is then constructed as $x = u(y,a)$. Later it will be shown that for any given function $u(y,a)$, there exists a unique function $U(u,a)$ satisfying (3.16), vice versa. The algorithm for computing $U(u,a)$ from a given $u(y,a)$, or for computing $u(y,a)$ from a given $U(u,a)$ will be derived later.

In practice, only the first few terms in the Taylor series of f (or u, v, g, U respectively) in a are important in stability analysis. However, it is convenient to develop the method for infinite power series in a even though none of functions f, u, v, g, and U will necessarily

have a convergent power series in a. This will present no difficulty as long as only a finite number of terms of the series are calculated.

Using (3.16), an algorithm for deriving (3.15) from (3.12) is given in the following Theorem.

Theorem 12 [15] Let $f(x,a)$, $g(y,a)$ and $U(u,a)$ in (3.12), (3.15) and (3.16) be expanded as follows:

$$f(x,a) = \sum_{m=0}^{\infty} f_m(x)\frac{a^m}{m!}, \tag{3.17}$$

$$g(y,a) = \sum_{m=0}^{\infty} g_m(y)\frac{a^m}{m!}, \tag{3.18}$$

$$U(u,a) = \sum_{m=0}^{\infty} U_m(u)\frac{a^m}{m!}. \tag{3.19}$$

The sequence $\{g_i\}$ in (3.18) satisfies

$$g_i = f_0^{(i)}, \tag{3.20}$$

where, for $i = 0, 1, 2, \cdots$, the sequence $f_{i-m}^{(m)}$ is defined recursively via

$$f_i^{(0)} = f_i \tag{3.21}$$

$$f_{i-m}^{(m)} = f_{i-m+1}^{(m-1)} + \sum_{j=0}^{i-m} \binom{i-m}{j} \left[U_j,\ f_{i-m-j}^{(m-1)}\right], \quad m = 1, 2, \cdots, i. \tag{3.22}$$

Here $\binom{i-m}{j} = \frac{(i-m)!}{j!(i-m-j)!}$ is the binomial coefficient and $[f,\ g](x) = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g$ is the Lie bracket. ◇◇◇

This algorithm can be visualized as follows,

$$
\begin{array}{lllllll}
f_0 = f_0^{(0)} = g_0 & \rightarrow & & & & & \\
 & \downarrow & & & & & \\
f_1 = f_1^{(0)} & \rightarrow & f_0^{(1)} = g_1 & & & & \\
 & & \downarrow & & & & \\
f_2 = f_2^{(0)} & \rightarrow & f_1^{(1)} & \rightarrow & f_0^{(2)} = g_2 & & \\
 & & \downarrow & & \downarrow & & \\
f_3 = f_3^{(0)} & \rightarrow & f_2^{(1)} & \rightarrow & f_1^{(2)} & \rightarrow & f_0^{(3)} = g_3 \\
 & & \downarrow & & \downarrow & & \downarrow \\
f_4 = f_4^{(0)} & \rightarrow & f_3^{(1)} & \rightarrow & f_2^{(2)} & \rightarrow & f_1^{(3)} \quad \rightarrow \quad f_0^{(4)} = g_4 \\
 & & \downarrow & & \downarrow & & \downarrow \qquad\qquad \downarrow
\end{array}
$$

The dependency between $\{U_m\}$, $\{f_m\}$, $\{g_m\}$ is clear from this graph.

The first few elements in the sequence $\{g_m\}$ can be easily written out

$$
\begin{aligned}
g_0 &= f_0, \\
g_1 &= f_1 + [U_0, f_0] = f_1 - [f_0, U_0], \\
g_2 &= f_2 + [U_0, f_1] + [U_0, g_1] - [f_0, U_1], \\
g_3 &= f_3 + [U_0, f_2] + 2[U_1, f_1] + [U_0, f_2 + [U_0, f_1] + [U_1, f_0]] \\
&\quad + [U_1, g_1] + [U_0, g_2] - [f_0, U_2].
\end{aligned}
$$

In general, it is easy to verify that

$$
g_m = \Gamma_m - [f_0, U_{m-1}] \tag{3.23}
$$

where Γ_m only depends on $f_0, f_1, \cdots, f_m$, and $U_0, \cdots, U_{m-2}$. The similarity between (3.7) and (3.23) should be observed.

The next Theorem is useful for deriving an explicit expression of $u(y,a)$ when the sequence $\{U_m\}$ is known.

Theorem 13 [15] Assume $u(y,a)$ is the solution to the PDE (3.16), and

$$p(x,a) = \sum_{m=0}^{\infty} p_m(x)\frac{a^m}{m!}. \tag{3.24}$$

Let

$$q(y,a) = p(u(y,a),a) = \sum_{m=0}^{\infty} q_m(y)\frac{a^m}{m!}.) \tag{3.25}$$

The sequence $\{q_m\}$ in (3.25) satisfies

$$q_i = p_0^{(i)}, \tag{3.26}$$

where, for $i = 0,1,2,\cdots$, the sequence $p_{i-m}^{(m)}$ is defined recursively via:

$$p_i^{(0)} = p_i \tag{3.27}$$

$$p_{i-m}^{(m)} = p_{i-m+1}^{(m-1)} + \sum_{j=0}^{i-m} \binom{i-m}{j} p_{i-m-j}^{(m-1)} * U_j,\ m = 1,2,\cdots,i. \tag{3.28}$$

Here $\binom{i-m}{j} = \frac{(i-m)!}{j![(i-m-j)]!}$ is the binomial coefficient and $f * g(x) = \frac{\partial f(x)}{\partial x} g(x)$ is the Lie derivative. ◇◇◇

Similarly, the algorithm can also be visualized as follows:

$$
\begin{array}{ccccccccc}
p_0 = p_0^{(0)} = q_0 & \rightarrow & & & & & & & \\
\downarrow & & & & & & & & \\
p_1 = p_1^{(0)} & \rightarrow & p_0^{(1)} = q_1 & & & & & & \\
\downarrow & & & & & & & & \\
p_2 = p_2^{(0)} & \rightarrow & p_1^{(1)} & \rightarrow & p_0^{(2)} = q_2 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
p_3 = p_3^{(0)} & \rightarrow & p_2^{(1)} & \rightarrow & p_1^{(2)} & \rightarrow & p_0^{(3)} = q_3 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
p_4 = p_4^{(0)} & \rightarrow & p_3^{(1)} & \rightarrow & p_2^{(2)} & \rightarrow & p_1^{(3)} & \rightarrow & p_0^{(4)} = q_4 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}
$$

The dependency between $\{U_m\}$, $\{p_m\}$, $\{q_m\}$ is observed clearly from this graph.

A procedure for deriving the explicit expression of $u(y,a)$ is given as follows.

Procedure 1: Applying Theorem 13 with $p(x,a) = U(x,a)$ yields an expansion for $q(y,a) = U(u(y,a),a) = \sum_{m=0}^{\infty} q_m(y)\frac{a^m}{m!}$. Substituting $u(y,a)$ and $U(u(y,a),a)$ into the left and right-hand sides of the PDE (3.16), respectively, gives

$$u_{m+1} = q_m, \quad m = 0, 1, 2, 3, \cdots.$$

◊◊◊

The first few elements in the sequence $\{q_m\}$ can also be easily written out as follows:

$$u_1(y) = q_0(y) = U_0(y)$$

$$
\begin{aligned}
u_2(y) &= q_1(y) = U_1(y) + \frac{\partial U_0(y)}{\partial y} U_0(y) \\
u_3(y) &= q_2(y) = U_2(y) + \frac{\partial U_1(y)}{\partial y} U_0(y) + \frac{\partial U_0(y)}{\partial y} U_1(y) + \frac{\partial q_1(y)}{\partial y} U_0(y) \\
u_4(y) &= q_3(y) = U_3(y) + \frac{\partial U_2(y)}{\partial y} U_0(y) + 2\frac{\partial U_1(y)}{\partial y} U_1(y) \\
&\quad + \frac{\partial}{\partial y}\left(U_2(y) + \frac{\partial U_1(y)}{\partial y} U_0(y) + \frac{\partial U_0(y)}{\partial y} U_1(y) \right) U_0(y) \\
&\quad + \frac{\partial q_1(y)}{\partial y} U_1(y) + \frac{\partial q_2(y)}{\partial y} U_0(y) + \frac{\partial U_0(y)}{\partial y} U_2(y).
\end{aligned}
$$

It is also easy to verify that u_m has the general form:

$$
u_m = U_{m-1} + \Lambda_m, \; m = 1, 2, 3, \cdots, \tag{3.29}
$$

where Λ_m only depends on U_0, U_1, $\cdots$, U_{m-2}. Therefore, we can conclude that for a given sequence $\{u_m\}$, there exists a unique sequence $\{U_m\}$ satisfying the PDE (3.16), vice-versa.

Later, we will need the following Lemma which is proved in [15].

Lemma 1 [15] If each f_m is a homogeneous polynomial map of degree of $m+1$, and each U_m is a homogeneous polynomial map of degree of $m+2$, then g_m, v_m, and u_m are homogeneous polynomial maps of degree of $m+1$. ◊◊◊

3.2.2 A Procedure for Computation of Normal Forms

We have seen that the normal form can be derived using a sequence of *near identity transformations*. This computation method is very inefficient even it is good for demonstration of the concept and structure of normal forms. In additional, this method is not appropriate for implementation using a symbolic computation package such as *Maple* or *Mathematica*. Alternatively, the algorithms introduced in Theorem 12 and 13, with slight modifications, can be used to derive normal forms and the associated transformations.

Assume the original nonlinear dynamical system is given by

$$\dot{x} = f(x) = \sum_{m=0}^{\infty} f^{m+1}(x)\frac{1}{m!}, \quad x \in R^n. \tag{3.30}$$

where $f^1(x) = Ax$ with $A = Df(0)$, $f^m \in H(n,n,m)$. Let the sequence of decompositions

$$H(n,n,m) = R(n,n,m) \oplus C(n,n,m)$$

be chosen. We need to obtain the transformations

$$x = u(y) = \sum_{m=0}^{\infty} u^{m+1}(y)\frac{1}{m!}, \quad y \in R^n, \tag{3.31}$$

where $u^1(y) = y$ and $u^m \in H(n,n,m)$, as well as

$$y = v(x) = \sum_{m=0}^{\infty} v^{m+1}(x)\frac{1}{m!} \tag{3.32}$$

where $v^1(x) = x$ and $v^m \in H(n,n,m)$, so that the dynamics for y are given by:

$$\dot{y} = g(y) \triangleq \frac{\partial v(x)}{\partial x} f(x)|_{x=u(y)} = \sum_{m=0}^{\infty} g^{m+1}(y)\frac{1}{m!}, \tag{3.33}$$

where $g^1(y) = Ay$, and $g^m \in C(n,n,m)$. First of all, we introduce an artificial parameter by scaling the variables according to $x \longmapsto ax$, $y \longmapsto ay$, and obtain

$$\dot{x} = f(ax)/a \triangleq f(x,a) = \sum_{m=0}^{\infty} f^{m+1}(x)\frac{a^m}{m!}, \tag{3.34}$$

$$x = u(ay)/a \triangleq u(y,a) = \sum_{m=0}^{\infty} u^{m+1}(y)\frac{a^m}{m!}, \tag{3.35}$$

$$y = v(ax)/a \triangleq v(x,a) = \sum_{m=0}^{\infty} v^{m+1}(x)\frac{a^m}{m!}, \tag{3.36}$$

$$\dot{y} = g(ay)/a \triangleq g(y,a) = \sum_{m=0}^{\infty} g^{m+1}(y)\frac{a^m}{m!}. \tag{3.37}$$

The PDE will be, according Lemma 1,

$$\frac{\partial u}{\partial a} = U(u,a) = \sum_{m=0}^{\infty} U^{m+2}(u)\frac{a^m}{m!}, \quad u(y,0) = y. \tag{3.38}$$

For the purpose of computing normal forms, $v(\cdot)$ in (3.32) may be taken to be the normal form transformation and the dynamics (3.33) would be the resulting normal form. Thus the process of computing the normal form dynamics can be accomplished recursively with successive computation of $g^i(\cdot)$. This procedure is summarized as follows:

Procedure 2: Take f_m, g_m and U_{m-1} in Theorem 12 to be f^{m+1}, g^{m+1}, and U^{m+1} in equations (3.34), (3.37), and (3.38), respectively. Beginning at Step 1, with $m = 1$ and $g^1 = f^1$, each pair U^m and g^m is determined iteratively. At Step m, a parameterized form for $U^m \in H(n,n,m)$ is assumed. Note that $g^1, \cdots, g^{m-1}$ and $U^2, \cdots, U^{m-1}$ are determined from previous steps. The homogeneous mapping g^m is then computed using the algorithm in Theorem 12. Restricting g^m to $C(n,n,m)$ will fix some coefficients in U^m. Other coefficients in U^m are free to be chosen. These free coefficients are simply set to zero in this dissertation. Once all of the coefficients in U^m are determined, both u^m and g^m are also determined. ◇◇◇

3.3 Further Discussion about Normal Forms

We have seen that (with $f_0(x) = Ax$, $g_m = g^{m+1}$, $U_{m-1} = U^{m+1}$)

$$g_m = \Gamma_m + [U_{m-1}, f_0] = \Gamma_m - [f_0, U_{m-1}]$$

$$= \Gamma_m - L_A^{m+1} U_{m-1}, \tag{3.39}$$

where Γ_m is the terms of degree $m+1$ after transformation $x = \sum_{k=0}^{m-1} u_k(y)/k!$ has been applied to the original nonlinear dynamical system. For the linear operator L_A^{m+1}, the space $H(n,n,m+1)$ has the following two decompositions (one in the domain, the other in the range):

$$\begin{aligned} H(n,n,m+1) &= H_1^D(n,n,m+1) \oplus H_2^D(n,n,m+1) \\ &= H_1^R(n,n,m+1) \oplus H_2^R(n,n,m+1), \end{aligned}$$

where $H_2^D(n,n,m+1)$ is the null space of L_A^{m+1}, $H_1^R(n,n,m+1)$ is the range of L_A^{m+1}, i.e.,

$$\begin{aligned} H_1^R(n,n,m+1) &= R(L_A^{m+1}) = R(n,n,m+1), \\ H_2^R(n,n,m+1) &= N(L_{A^*}^{m+1}) = C(n,n,m+1), \\ H_1^D(n,n,m+1) &= R(L_{A^*}^{m+1}), \\ H_2^D(n,n,m+1) &= N(L_A^{m+1}). \end{aligned}$$

Let P_1^D be the projection of $H(n,n,m+1)$ onto $H_1^D(n,n,m+1)$ along $H_2^D(n,n,m+1)$. Similarly, let P_1^R be the projections of $H(n,n,m+1)$ onto $H_1^R(n,n,m+1)$ along $H_2^R(n,n,m+1)$. Let $\left(L_A^{m+1}\right)^{-1}$ be the inverse of L_A^{m+1} taking $H_1^R(n,n,m+1)$ to $H_1^D(n,n,m+1)$. A rigorous definition of the normal form is given as follows.

Definition 4 [15] Let (3.30) be the original nonlinear dynamical system. Equation (3.33) is a normal form for (3.30) relative to the matrix A and projections P_1^D, P_1^R if the elements

g_m, U_m *satisfy*

$$\begin{aligned} g_m(y) &= \left(I - P_1^R\right)\Gamma_m(y) \\ U_{m-1}(u) &= \left(L_A^{m+1}\right)^{-1} P_1^R\,\Gamma_m(y) \end{aligned}$$

where Γ_m *is defined in the equation (3.39).* ◇◇◇

Since $L_A^m : H(n,n,m) \to H(n,n,m)$ is a linear operator, it can be represented using a matrix $\widetilde{L}_A^m$ of appropriate dimension. It is also desirable to know the eigenvalues of this operator. Remember that a complex number Λ is a eigenvalue of L_A^m if and only if there exists a nonzero $P^m \in H(n,n,m)$ (possibly over complex) such that $L_A^m P^m(x) = \Lambda P^m(x)$.

The following theorems are proved in [15].

Theorem 14 [15] Let an $n \times n$ *matrix* T *be invertible and* $B = T^{-1}AT$. *The eigenvalues of the linear operators* L_A^m *and* L_B^m *are identical, i.e.,*

$$\sigma(L_A^m) = \sigma(L_B^m).$$

In other words, the spectrum of the linear operator L_A^m *is invariant under a linear change of coordinates.* ◇◇◇

Theorem 15 [15] Let the eigenvalues of a given $n \times n$ *matrix* A *be* $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *then*

$$\sigma(L_A^m) = \left\{ \sum_{i=1}^{n} \lambda_i \alpha_i - \lambda_j \,|\, j = 1,2,3,\cdots n; \sum_{i=1}^{n} \alpha_i = m; \alpha_i \geq 0 \right\} \tag{3.40}$$

◇◇◇

We now study how to derive the matrix $\tilde{L}_A^m$. $H(n,n,m)$ is a linear space. The dimension of $H(n,n,m)$ is equal to $n\frac{(n+m-1)!}{m!(n-1)!}$. The basis for $H(n,n,m)$ can be determined as follows. As we have seen before, the set of monomials $\{x^\alpha \mid |\alpha| = m\}$ is a basis for $H(n,1,m)$, and dimension of $H(n,1,m)$ is $\binom{n+m-1}{m}$. Let $\{e_1, \cdots, e_n\}$ be the natural basis for R^n, i.e. the j^{th} entry of the column vector e_j is one, while all other entries are zero. A basis for the space $H(n,n,m)$ is given by

$$\{x^\alpha e_j \mid |\alpha| = m, 1 \le j \le n\}, \tag{3.41}$$

with $d_m = \dim H(n,n,m) = n \cdot \binom{n+m-1}{m}$. We order the basis elements (described in (3.41)) of $H(n,n,m)$ in the reverse lexicographic order, i.e., $x^\alpha e_i$ proceeds $x^\beta e_j$ if and only if the first nonzero entry in the row vector $[i-j,\ \alpha_1-\beta_1,\ \cdots,\ \alpha_n-\beta_n]$ is positive. Therefore, a correspondence between the basis elements of R^{d_m} and $H(n,n,m)$ can be established. When both n and m are big, finding and ordering the basis for $H(n,n,m)$ is not a easy job. Fortunately, we can use symbolic computation software such as *Maple* to generate the basis.

Example 11 *(Derivation of Basis)*

The basis for $H(2,1,2)$ is $\{x^2, xy, y^2\}$ in reverse lexicographic order and $\dim H(2,1,2) = 3$. The basis for $H(2,2,2)$ in the reverse lexicographic order is

$$\left\{\begin{pmatrix}0\\x^2\end{pmatrix}, \begin{pmatrix}0\\xy\end{pmatrix}, \begin{pmatrix}0\\y^2\end{pmatrix}, \begin{pmatrix}x^2\\0\end{pmatrix}, \begin{pmatrix}xy\\0\end{pmatrix}, \begin{pmatrix}y^2\\0\end{pmatrix}\right\}.$$

and $\dim H(2,2,2) = 6$. ◇◇◇

Once the basis for a linear space is fixed, the matrix representation of the linear operator L_A^m can be derived by applying it to each basis element of $H(n, n, m)$.

Example 12 (Derivation of the Matrix Representation of the Linear Operators L_A^m)

For $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with a pair of critical eigenvalues, $\pm j$, the matrix representation for the operators L_A^2, L_A^3 can be expressed by

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -2 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -3 & 0 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The resulting eigenvalue sets are $\{3j, -3j, j, -j, j, -j\}$ and $\{0, 0, 4j, -4j, 2j, -2j, 2j, -2j\}$ respectively. ◇◇◇

3.4 Summary

In this chapter, normal form theory was introduced. The concept and structure of normal forms were first illustrated using near identity change of coordinates. An efficient algorithm for the computation of the normal form was presented. The problem of simplification of nonlinear dynamics could be reduced to the study of the linear operator L_A^k.

CHAPTER 4
RELATIONSHIP BETWEEN CENTER MANIFOLD AND NORMAL FORM THEORIES

As we will see, center manifold and normal form methods are two major model reduction techniques in study of critical nonlinear systems. In this chapter, the relationship between these two methods will be explored. As we already knew, center manifolds for a given critical nonlinear system are not unique. However, it will be proved in Section 4.1 that all center manifolds for a given critical nonlinear system are equivalent in some sense. In Section 4.2, normal forms of critical nonlinear systems are studied. The relationship between center manifold and normal form theories is presented in Section 4.3. Section 4.4 includes an example to demonstrate the equivalence of center manifold, normal forms of critical nonlinear systems, and the relationship between center manifold and normal form theories. A brief summary in section 4.5 concludes this chapter.

4.1 Equivalence of Center Manifolds

As we mentioned before, center manifolds need not to be unique for a given nonlinear system. In this section, we will prove that all center manifolds for a given C^k nonlinear system are k-jet equivalent. First of all, the concept of k-jet equivalence is defined.

Definition 5 [17] Let $f, g : D \subset R^n \to R^m$ be two C^k maps with $k \geq 1$, being an integer. The maps f and g are k-jet equivalent on a set $\Omega \subset D$ if and only if their partial derivatives

up to order k coincide on D:

$$\left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{k_n} (f-g)(x_0) = 0$$

for all $x_0 \in D$, and non-negative integers $[k_1, k_2, \cdots, k_n]$, with $0 \leq \sum_{i=1}^{n} k_i \leq k$. In the other words, the Taylor series expansions of f and g on D coincide up to degree k. $\Diamond\Diamond\Diamond$

The following lemma is needed later. The proof of this lemma is found in [15].

Lemma 2 *[15] For a given $n \times n$ matrix A and a given $m \times m$ matrix B, with eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, and $\{\mu_1, \cdots, \mu_m\}$, respectively, define the linear operator $L^r_{A,B} : H(m,n,r) \to H(m,n,r)$ via*

$$L^r_{A,B}(P^r(x)) = \frac{\partial P^r(x)}{\partial x} Bx - AP^r(x), \tag{4.1}$$

where $P^r(x) \in H(m,n,r)$. Then the spectrum of $L^r_{A,B}$ is given by

$$\sigma(L^r_{A,B}) = \left\{ \sum_{i=1}^{m} q_i \mu_i - \lambda_j \mid j = 1, 2, \cdots n; \sum_{i=1}^{m} q_i = r \quad \text{with} \quad q_i \geq 0 \right\}. \tag{4.2}$$

$\Diamond\Diamond\Diamond$

Consider a critical C^k nonlinear system defined on some neighborhood Ω of the origin of R^n, $(k \geq 2)$. Without loss of generality, this system may be described as follows;

$$\begin{aligned} \dot{x}_c &= A^c x_c + f_c(x_c, x_s) \\ &= A^c x_c + f_c^2(x_c, x_s) + \cdots + f_c^k(x_c, x_s) + O(|x|^{k+1}), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \dot{x}_s &= A^s x_s + f_s(x_c, x_s) \\ &= A^s x_s + f_s^2(x_c, x_s) + \cdots + f_s^k(x_c, x_s) + O(|x|^{k+1}), \end{aligned} \tag{4.4}$$

where $x_c \in R^{n_c}$, $x_s \in R^{n_s}$, $n_c + n_s = n$, $f_c^i \in H(n, n_c, i)$, and $f_s^i \in H(n, n_s, i)$. From center manifold theory, there exists a C^k center manifold,

$$x_s = h(x_c) = h^2(x_c) + h^3(x_c) + \cdots + h^k(x_c) + O(|x_c|^{k+1}), \tag{4.5}$$

for the system (4.3), (4.4) on some neighborhood $\Omega' \subset R^{n_c}$, such that the local stability of the system (4.3), (4.4) is equivalent to the local stability of the subsystem

$$\dot{x}_c = A^c x_c + f_c(x_c, h(x_c)). \tag{4.6}$$

Theorem 16 *All center manifolds for (4.3), (4.4) are $k-$jet equivalent on Ω'.* ◇◇◇

Proof: A parameterized function $x_s = h(x_c) = h^2(x_c) + h^3(x_c) + \cdots + h^k(x_c) + O(|x_c|^{k+1})$ is assumed first. If $x_s = h(x_c)$ is a center manifold for (4.3), (4.4), by definition, it satisfies following Partial Differential Equation (PDE),

$$\frac{\partial h(x_c)}{\partial x_c}(A^c x_c + f_c(x_c, h(x_c))) = A^s h(x_c) + f_s(x_c, h(x_c)). \tag{4.7}$$

More specifically,

$$\frac{\partial (h^2(x_c) + h^3(x_c) + \cdots)}{\partial x_c}\left(A^c x_c + f_c^2(x_c, h(x_c)) + f_c^3(x_c, h(x_c)) + \cdots\right) \tag{4.8}$$

$$= A^s h(x_c) + f_s^2(x_c, h(x_c)) + f_s^3(x_c, h(x_c)) + \cdots. \tag{4.9}$$

It is clear that the homogeneous terms of degree $r \leq k$ on the left-hand side of the equation (4.9) only depend on $h^2, \cdots, h^r$ and A^c, $f_c^2, \cdots, f_c^r$. Similarly, the homogeneous terms of degree $r \leq k$ on the right-hand side of the equation (4.9) only depends on $h^2, \cdots, h^r$

and A^s, f_s^2, $\cdots$, f_s^r. With a little work, we can show that h^r satisfies the following PDE,

$$\frac{\partial h^r(x_c)}{\partial x_c} A^c x_c = A^s h^r(x_c) + \Pi^r(x_c) \tag{4.10}$$

for some function $\Pi^r(x_c)$ which depends on $h^2, \cdots, h^{r-1}$, $f_c^2, \cdots, f_c^r$ and $f_s^2, \cdots, f_s^r$. Notice that all eigenvalues of A^c have zero real parts, while all of the eigenvalues of A^s have negative real parts. From the Lemma 2, the linear operator $L^r_{A^s, A^c}$ does not have any zero eigenvalues. This means that

$$L^r_{A^s, A^c}(h^r(x_c)) = \frac{\partial h^r(x_c)}{\partial x_c} A^c x_c - A^s h^r(x_c) = \Pi^r(x_c), \tag{4.11}$$

has an unique solution $h^r(x_c)$ for any function $\Pi^r(x_c) \in H(n_c, n_s, r)$ where $r = 2, 3, \cdots, k$.

◇◇◇

Example 13 (Equivalence of Center Manifolds)

We have shown that for any real number c_1, c_2,

$$x_s = h(x_c, c_1, c_2) = \begin{cases} c_1 \exp(-\frac{1}{2} x_c^{-2}), & x_c > 0 \\ 0 & x_c = 0 \\ c_2 \exp(-\frac{1}{2} x_c^{-2}), & x_c < 0 \end{cases}$$

is a center manifold for the system

$$\dot{x}_c = -x_c^3 \tag{4.12}$$

$$\dot{x}_s = -x_s. \tag{4.13}$$

It is very straightforward to show that $h^k(x_c, c_1, c_2) = 0$ for any integer $k \geq 0$. $\diamond\diamond\diamond$

4.2 Normal Forms of Critical Nonlinear Systems

Among other applications, normal forms are capable of bringing out the essential features of nonlinear systems. In this section, we use normal form technique to simplify description of critical nonlinear systems. As we proceed, the reason the normal form technique is called a model reduction method will become evident.

First, let us consider an analytic critical nonlinear systems over complex of form

$$\dot{x} = Ax + f(x) = Ax + f^2(x) + f^3(x) + \cdots, \quad x \in C^n. \tag{4.14}$$

Definition 6 [4] The eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of A are resonant if, for at least one value of r, $1 \leq r \leq n$,

$$\lambda_r = \sum_{i=1}^{n} q_i \lambda_i$$

holds for some non-negative integers $q \stackrel{def}{=} [q_1, \cdots, q_n]$ satisfying $\sum_{i=1}^{n} q_i \geq 2$. $\diamond\diamond\diamond$

Remark 6 The eigenvalues of any real matrix with critical eigenvalues are necessarily resonant.

Let $e = \{e_1, \cdots e_n\}$ be the standard basis for R^n, so that the i^{th} entry of e_i is 1 while all other entries are zero. Let $x^q = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ be a scalar monomial.

Definition 7 [4] The vector monomial $x^q e_j$ is resonant with respect to an upper triangular matrix A (possibly complex) with (ordered) eigenvalues $\lambda_1, \ldots, \lambda_n$, if, for those non-negative integers $[q_1, \ldots, q_n]$ indexing x^q,

$$\lambda_j - \sum_{i=1}^{n} q_i \lambda_i = 0. \tag{4.15}$$

The integer $r = \sum_{i=1}^{n} q_i, \geq 2$ *is called the degree of the resonant monomial.* $\diamond\diamond\diamond$

The following two lemmas are proved in [4].

Lemma 3 [4] If A is an upper triangular matrix with diagonal elements $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *then a normal form up to degree k for the system (4.14) is given by*

$$\dot{\overline{x}} = A\overline{x} + \overline{f}^2(\overline{x}) + \cdots + \overline{f}^r(\overline{x}) + O(|\overline{x}|^{r+1}) \tag{4.16}$$

where $\overline{f}^i(\overline{x})$, $i = 2, \cdots, k$, *is made up of linear combinations of all resonant monomials.* $\diamond\diamond\diamond$

Remark 7 The actual description of $\overline{f}^i(\overline{x})$ *is not unique. It depends on A and other factors. On the other hand, not all resonant monomials of degree i are included in* $\overline{f}^i(\overline{x})$. $\diamond\diamond\diamond$

Lemma 4 [4] Assuming that the following equation

$$\dot{\overline{x}} = A\overline{x} + \overline{f}^2(\overline{x}) + \cdots + \overline{f}^k(\overline{x}) + O\left(|x|^{k+1}\right) \tag{4.17}$$

represent a normal form up to degree k with respect to A. Then after a linear change of coordinates $\overline{x} = T\overline{y}$, *the resulting equation*

$$\dot{\overline{y}} = B\overline{y} + \overline{g}^2(\overline{y}) + \cdots + \overline{g}^k(\overline{y}) + O\left(|\overline{y}|^{k+1}\right), \tag{4.18}$$

will be a normal form up to degree k with respect to $B := T^{-1}AT$, *where* $\overline{g}^i(\overline{y}) = T^{-1}\overline{f}^i(T\overline{y})$, $i = 2, \cdots, k$. $\diamond\diamond\diamond$

Remark 8 In this dissertation, only real systems are considered. Some real matrices have complex eigenvalues so that their Jordan forms are complex matrices. After deriving the

structure of normal forms over the field of complex number, the structure over the field of real numbers will be can be determined easily. ◇◇◇

Consider a real critical nonlinear system as follows:

$$\begin{aligned}\dot{x}_c &= A^c x_c + f_c(x_c, x_s) \\ &= A^c x_c + f_c^2(x_c, x_s) + f_c^3(x_c, x_s) + \cdots, \quad x_c \in R^{n_c} \qquad (4.19)\\ \dot{x}_s &= A^s x_s + f_s(x_c, x_s) \\ &= A^s x_s + f_s^2(x_c, x_s) + f_s^3(x_c, x_s) + \cdots, \quad x_s \in R^{n_s}. \qquad (4.20)\end{aligned}$$

The following theorem describes the structure of normal forms for the system (4.19)-(4.20).

Theorem 17 All normal forms up to degree k for the system (4.19)-(4.20) have the following structure;

$$\begin{aligned}\dot{\bar{x}}_c &= A^c \bar{x}_c + \bar{f}_c(\bar{x}_c, \bar{x}_s) \\ &= A^c \bar{x}_c + \bar{f}_c^2(\bar{x}_c) + \bar{f}_c^3(\bar{x}_c) + \cdots + \bar{f}_c^k(\bar{x}_c) + O(|\bar{x}_c, \bar{x}_s|^{k+1}) \qquad (4.21)\\ \dot{\bar{x}}_s &= A^s \bar{x}_s + \bar{f}_s(\bar{x}_c, \bar{x}_s) \\ &= A^s \bar{x}_s + \bar{f}_s^2(\bar{x}_c, \bar{x}_s) + \bar{f}_s^3(\bar{x}_c, \bar{x}_s) + \cdots + \bar{f}_s^k(\bar{x}_c) + O(|\bar{x}_c, \bar{x}_s|^{k+1}) \qquad (4.22)\end{aligned}$$

with $\bar{f}_s^i(\bar{x}_c, \bar{x}_s = 0)$ *for* $i = 2, 3, 4, \cdots$.

Proof of Theorem 17: Let $A = \begin{pmatrix} A^c & 0 \\ 0 & A^s \end{pmatrix}$. There exists an invertible (possibly complex) matrix $T = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}$ such that $\Lambda = \begin{pmatrix} \Lambda^c & 0 \\ 0 & \Lambda^s \end{pmatrix} = T^{-1}AT = \begin{pmatrix} T_{11}^{-1} A^c T_{11} & 0 \\ 0 & T_{22}^{-1} A^s T_{22} \end{pmatrix}$

where Λ^c and Λ^s are upper triangular matrices with (possibly complex) diagonal elements being $\{\lambda_{c,1}, \cdots, \lambda_{c,n_c}\}$ and $\{\lambda_{s,1}, \cdots, \lambda_{s,n_s}\}$, respectively. After the linear transformation $x = Ty$ is applied to the system (4.19), (4.20), the dynamics for y become

$$\begin{aligned} \dot{y}_c &= \Lambda^c y_c + g_c(y_c, y_s) \\ &= \Lambda^c y_c + g_c^2(y_c, y_s) + g_c^3(y_c, y_s) + \cdots, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \dot{y}_s &= \Lambda^s y_s + g_s(y_c, y_s) \\ &= \Lambda^s y_s + g_s^2(y_c, y_s) + g_s^3(y_c, y_s) + \cdots. \end{aligned} \tag{4.24}$$

Assume that a normal form for the system (4.23)-(4.24) (up to order k) has the form:

$$\begin{aligned} \dot{\overline{y}}_c &= \Lambda^c \overline{y}_c + \overline{g}_c(\overline{y}_c, \overline{y}_s) \\ &= \Lambda^c \overline{y}_c + \overline{g}_c^2(\overline{y}_c, \overline{y}_s) + \overline{g}_c^3(\overline{y}_c, \overline{y}_s) + \cdots \end{aligned} \tag{4.25}$$

$$\begin{aligned} \dot{\overline{y}}_s &= \Lambda^s \overline{y}_s + \overline{g}_s(\overline{y}_c, \overline{y}_s) \\ &= \Lambda^s \overline{y}_s + \overline{g}_s^2(\overline{y}_c, \overline{y}_s) + \overline{g}_s^3(\overline{y}_c, \overline{y}_s) + \cdots. \end{aligned} \tag{4.26}$$

From Lemma 3, $\overline{g}_c^k(y_c, y_s)$ only contains monomials

$$y^q = y_{c,1}^{q_1} \cdots y_{c,n_c}^{q_{n_c}} \, y_{s,1}^{q_{n_c+1}} \cdots y_{s,n_s}^{q_n} \;, \; \sum_{i=1}^{n} q_i = k$$

such that

$$\lambda_{c,j} = \sum_{i=1}^{n_c} q_i \lambda_{c,i} + \sum_{i=1}^{n_s} q_{n_c+i} \lambda_{s,i} \;, \; 1 \leq j \leq n_c. \tag{4.27}$$

Recall that the real parts of $\lambda_{c,i}$ (or $\lambda_{s,i}$) are zero (or negatively respectively). To make equation (4.27) hold, it is necessary to have $q_{n_c+1} = \cdots = q_n = 0$, hence $\overline{g}_c^k(y_c, y_s) = \overline{g}_c^k(y_c)$.

Similarly, $\overline{g}_s^k(y_c, y_s)$ only contains monomials

$$y^q = y_{c,1}^{q_1} \cdots y_{c,n_c}^{q_{n_c}} y_{s,1}^{q_{n_c+1}} \cdots y_{s,n_s}^{q_n}, \ \sum_{i=1}^{n} q_i = k$$

such that

$$\lambda_{s,j} = \sum_{i=1}^{n_c} q_i \lambda_{c,i} + \sum_{i=1}^{n_s} q_{n_c+i} \lambda_{s,i}, \ 1 \le j \le n_s. \tag{4.28}$$

Therefore, it is necessary for $\sum_{i=n_c+1}^{n} q_i \neq 0$, hence $\overline{g}_s^k(y_c, y_s = 0) = 0$. The linear transformation $\overline{x} = T\overline{y}$ applied to (4.25), (4.26) will yield the system (4.21), (4.22). ◇◇◇

4.3 Relationship

The main results of this section show the relationship between center manifold and normal form theory. We first derive center manifolds for the system (4.21), (4.22).

Theorem 18 *Any center manifold for the system (4.21), (4.22) satisfies* $\overline{x}_s = O(|\overline{x}_c|^{k+1})$. *In other words, the k-jet equivalent of any center manifold* $\overline{h}(\overline{x}_c) = \overline{h}^2(\overline{x}_c) + \overline{h}^3(\overline{x}_c) + \cdots + \overline{h}^k(\overline{x}_c) + O(|\overline{x}_c|^{k+1})$ *must satisfy* $\overline{h}^i(\overline{x}_c) = 0$, *for* $i = 2, \ldots, k$. ◇◇◇

The proof of Theorem 18 is very similar to that of Theorem 16, except in this case $\Pi^i(\cdot) = 0$, for $i = 2, \cdots, k$ in the PDE (4.7). With this theorem, the dynamics for the system (4.21), (4.22) along any center manifold $\overline{x}_s = \overline{h}(\overline{x}_c)$, up to order k, satisfy

$$\begin{aligned} \dot{\overline{x}}_c &= A^c \overline{x}_c + \overline{f}_c(\overline{x}_c, \overline{h}(\overline{x}_c)) \\ &= A^c \overline{x}_c + \overline{f}_c^2(\overline{x}_c) + \cdots + \overline{f}_c^k(\overline{x}_c). \end{aligned}$$

Therefore, the stability of the system (4.21), (4.22) is determined by the stability of the subsystem (4.21). That is the exact reason the normal form method is called a model reduction technique.

The relationship between the two subsystems (4.6) and (4.21) (up to degree k) is given in the following Theorem.

Theorem 19 One of the approximate normal forms of the subsystem (4.6), (up to order k), is given by (4.21). ◇◇◇

To derive a normal form up to degree k, the transformation of form

$$x_c = \overline{x}_c + u_c^2(\overline{x}_c, \overline{x}_s) + \cdots + u_c^k(\overline{x}_c, \overline{x}_s) \tag{4.29}$$

$$x_s = \overline{x}_s + u_s^2(\overline{x}_c, \overline{x}_s) + \cdots + u_s^k(\overline{x}_c, \overline{x}_s). \tag{4.30}$$

is sufficient. The center manifolds for the system , as well as the transformation between (4.6) and (4.21) can be derived from equations (4.29), (4.30).

Theorem 20 The transformation $x_c = \overline{x}_c + \psi^2(\overline{x}_c) + \cdots + \psi^k(\overline{x}_c)$ taking (4.6) to (4.21) is obtained by setting $\overline{x}_s = 0$ in equation (4.29). Any center manifold $x_s = h(x_c)$ (up to degree k) can be derived from equations (4.29), (4.30), by setting $\overline{x}_s = 0$. ◇◇◇

4.4 A Example

Example 14 (A CASE 2 system)

Consider following system:

$$\dot{x} = -y + xz^2 + xyz + x^2 + z^2, \tag{4.31}$$

$$\dot{y} = x + z^2 + x^2 + xz^2, \tag{4.32}$$

$$\dot{z} = -z + xy + z^2, \tag{4.33}$$

with $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. The center manifold (up to degree 3) for the system (4.31)-(4.33) is

$$z = h(x,y) = -\frac{x^2}{5} + \frac{xy}{5} + \frac{y^2}{5} + \frac{x^3}{5} + O(\|x,y\|^4). \tag{4.34}$$

The dynamics (up to degree 3) along the center manifold is

$$\dot{x} = -y + x^2 + O(\|x,y\|^4) \tag{4.35}$$

$$\dot{y} = x + x^2 + O(\|x,y\|^4). \tag{4.36}$$

The normal form up to degree 3 for the system (4.35)-(4.36) is given by

$$\dot{\bar{x}} = -\bar{y} - \frac{1}{12}\left(\bar{x}^2 + \bar{y}^2\right)(3\,\bar{x} - 7\,\bar{y}) \tag{4.37}$$

$$\dot{\bar{y}} = \bar{x} - \frac{1}{12}\left(\bar{x}^2 + \bar{y}^2\right)(7\,\bar{x} + 3\,\bar{y}) \tag{4.38}$$

$$\dot{\bar{z}} = -\bar{z}. \tag{4.39}$$

The center manifold for the system (4.37)-(4.39) is, up to degree 3,

$$\bar{z} = 0 \tag{4.40}$$

The backward transformations which transforms (4.31)-(4.33) to (4.37)-(4.39) is

$$x = \bar{x} + -\frac{\bar{x}^2}{3} + \frac{2\,\bar{x}\bar{y}}{3} - \frac{2\,\bar{y}^2}{3} - \frac{3\,\bar{z}^2}{5} - \frac{7\,\bar{x}^3}{36}$$

$$
\begin{aligned}
&+\frac{6\,\bar{y}^2\bar{z}}{25}+\frac{4\,\bar{z}^3}{5}-\frac{\bar{y}\bar{z}^2}{20}-\frac{\bar{x}^2\bar{y}}{4}-\frac{43\,\bar{x}\bar{y}\bar{z}}{50}\\
&-\frac{2\,\bar{y}^3}{3}+\frac{\bar{x}\bar{z}^2}{10}-\frac{6\,\bar{x}^2\bar{z}}{25}+\frac{2\,\bar{x}\bar{y}^2}{9} \qquad (4.41)\\
y \;=\;& \bar{y}+\frac{\bar{x}^2}{3}+\frac{2\,\bar{x}\bar{y}}{3}+\frac{2\,\bar{y}^2}{3}-\frac{\bar{z}^2}{5}\frac{2\,\bar{z}^3}{5}\\
&-\frac{21\,\bar{x}\bar{y}\bar{z}}{50}-\frac{\bar{x}^3}{6}+\frac{\bar{x}\bar{z}^2}{20}-\frac{11\,\bar{y}^2\bar{z}}{50}\\
&-\frac{13\,\bar{y}^3}{36}-\frac{5\,\bar{x}^2\bar{y}}{18}+\frac{5\,\bar{x}\bar{y}^2}{12}+\frac{11\,\bar{x}^2\bar{z}}{50} \qquad (4.42)\\
z \;=\;& \bar{z}-\frac{\bar{x}^2}{5}+\frac{\bar{x}\bar{y}}{5}+\frac{\bar{y}^2}{5}-\bar{z}^2+\frac{2\,\bar{x}^3}{5}\\
&+\frac{\bar{y}\bar{z}^2}{5}+\frac{2\,\bar{x}\bar{z}^2}{5}+\frac{2\,\bar{y}^2\bar{z}}{5}-\frac{2\,\bar{x}\bar{y}\bar{z}}{5}\\
&+\frac{2\,\bar{y}^3}{15}-\frac{\bar{x}^2\bar{y}}{15}+\frac{4\,\bar{x}\bar{y}^2}{5}+\frac{\bar{x}^2\bar{z}}{5}+\bar{z}^3 \qquad (4.43)
\end{aligned}
$$

The forward transformation is

$$
\begin{aligned}
\bar{x} \;=\;& x+\frac{x^2}{3}-\frac{2\,xy}{3}+\frac{2\,y^2}{3}+\frac{3\,z^2}{5}+\frac{23\,x^3}{36}\\
&-\frac{5\,x^2y}{12}+\frac{12\,x^2z}{25}+\frac{2\,xy^2}{9}+\frac{31\,xyz}{50}+\frac{xz^2}{6}\\
&-\frac{5\,x^2y}{12}+\frac{12\,x^2z}{25}+\frac{2\,xy^2}{9}+\frac{31\,xyz}{50}+\frac{xz^2}{6} \qquad (4.44)\\
\bar{y} \;=\;& y-\frac{x^2}{3}-\frac{2\,xy}{3}-\frac{2\,y^2}{3}+\frac{z^2}{5}+\frac{x^3}{6}\\
&+\frac{25\,x^2y}{18}-\frac{7\,x^2z}{50}+\frac{11\,xy^2}{12}+\frac{17\,xyz}{50}\\
&-\frac{7\,xz^2}{12}+\frac{29\,y^3}{36}+\frac{7\,y^2z}{50}-\frac{2\,yz^2}{3} \qquad (4.45)\\
\bar{z} \;=\;& z+\frac{x^2}{5}-\frac{xy}{5}-\frac{y^2}{5}+z^2-\frac{x^3}{5}+\frac{x^2z}{5}\\
&-\frac{xz^2}{5}-\frac{4\,y^2z}{5}-\frac{2\,yz^2}{5}+z^3 \qquad (4.46)
\end{aligned}
$$

The backward transformation which brings (4.35)-(4.36) to (4.37)-(4.38) is obtained from equations (4.41)-(4.42) by simply setting $\bar{z} = 0$, that is

$$\begin{aligned} x &= \bar{x} - \frac{\bar{x}^2}{3} + \frac{2\,\bar{x}\bar{y}}{3} - \frac{2\,\bar{y}^2}{3} - \frac{7\,\bar{x}^3}{36} - \frac{\bar{x}^2\bar{y}}{4} - \frac{2\,\bar{y}^3}{3} + \frac{2\,\bar{x}\bar{y}^2}{9} \\ y &= \bar{y} + \frac{\bar{x}^2}{3} + \frac{2\,\bar{x}\bar{y}}{3} + \frac{2\,\bar{y}^2}{3} - \frac{\bar{x}^3}{6} - \frac{13\,\bar{y}^3}{36} - \frac{5\,\bar{x}^2\bar{y}}{18} + \frac{5\,\bar{x}\bar{y}^2}{12}. \end{aligned}$$

The forward transformation which brings (4.35)-(4.36) to (4.37)-(4.38) can be derived in many different ways. First, the algorithm in the Chapter 3 can be used. Alternatively, it can be derived by solving equations (4.44)-(4.46) together with $\bar{z} = 0$. A small trick can be used as follows. From equation (4.46), we have

$$z = -\left(\frac{x^2}{5} - \frac{xy}{5} - \frac{y^2}{5} + z^2 - \frac{x^3}{5} + \frac{x^2z}{5} - \frac{xz^2}{5} - \frac{4\,y^2z}{5} - \frac{2\,yz^2}{5} + z^3\right). \tag{4.47}$$

Next, equation (4.47) is substituted recursively into equations (4.44)-(4.45) until z is only occurs in terms of degree 4 or higher. After truncating terms of degree 4 or higher, we obtain the forward transformation as follows:

$$\begin{aligned} \bar{x} &= x + \frac{x^2}{3} - \frac{2\,xy}{3} + \frac{2\,y^2}{3} - \frac{x^2y}{6} - \frac{5\,y^3}{12} + \frac{x^3}{18} - \frac{13\,xy^2}{36} \\ \bar{y} &= y - \frac{2\,xy}{3} - \frac{2\,y^2}{3} - \frac{x^2}{3} + \frac{29\,x^2y}{36} + \frac{2\,y^3}{9} + \frac{2\,xy^2}{3} - \frac{x^3}{12}. \end{aligned}$$

The center manifold (4.34) (up to degree 3) can be solved from equations (4.41)-(4.43) by simply setting $\bar{z} = 0$, or derived through recursive substitution using the equation (4.47).

4.5 Summary

In this chapter, the equivalence of center manifolds for a given critical nonlinear system is studied first. Next, the structure of normal forms for critical nonlinear systems is derived. The reason the normal form method is called a model reduction technique is explained. The relationship between center manifold and normal form theories is explored. Finally, a example is provided to illustrate this relationship.

CHAPTER 5
CONSTRUCTION OF LYAPUNOV FUNCTIONS – I

This chapter is concerned with the construction of Lyapunov functions of critical nonlinear dynamical systems if they are LAS. As we mentioned before, a necessary condition for a given nonlinear dynamical system $\dot{x} = f(x)$ to be LAS is that the eigenvalues of its Jacobian matrix $A = Df(0)$ are either stable or critical. If the Jacobian matrix possesses only stable eigenvalues, then a Lyapunov function for the corresponding linear system $\dot{x} = Ax$ is also a Lyapunov function for the nonlinear system itself. The only situation which needs further investigation is when the Jacobian matrix possesses stable eigenvalues and at least one critical eigenvalue. Therefore, only critical nonlinear systems will be considered in this chapter. In Section 5.1 we will present a procedure for constructing Lyapunov functions for general critical nonlinear systems. To apply this procedure, it is required that there exist a full normal form for the given critical nonlinear system. In Section 5.2 an explicit algorithm for construction of Lyapunov functions for nonlinear systems in two special cases is provided. This algorithm only requires an approximate normal form. Examples are given in Section 5.4, to demonstrate use of the algorithm along with some discussions. The major points in this chapter are summarized in Section 5.5.

5.1 General Procedure For Constructing Lyapunov Functions

Consider a critical nonlinear system described below:

$$\dot{x}_c = A^c x_c + f_c(x_c, x_s)$$

$$= A^c x_c + f_c^2(x_c, x_s) + f_c^3(x_c, x_s) + \cdots, \quad x_c \in R^{n_c}, \tag{5.1}$$

$$\dot{x}_s = A^s x_s + f_s(x_c, x_s)$$

$$= A^s x_s + f_s^2(x_c, x_s) + f_s^3(x_c, x_s) + \cdots, \quad x_s \in R^{n_s}. \tag{5.2}$$

If a full normal form for the system (5.1) and (5.2) exists, the normal form will have the following structure:

$$\dot{\bar{x}}_c = A^c \bar{x}_c + \bar{f}_c(\bar{x}_c)$$

$$= A^c \bar{x}_c + \bar{f}_c^2(\bar{x}_c) + \bar{f}_c^3(\bar{x}_c) + \cdots, \tag{5.3}$$

$$\dot{\bar{x}}_s = A^s \bar{x}_s + \bar{f}_s(\bar{x}_c, \bar{x}_s)$$

$$= A^s \bar{x}_s + \bar{f}_s^2(\bar{x}_c, \bar{x}_s) + \bar{f}_s^3(\bar{x}_c, \bar{x}_s) + \cdots, \tag{5.4}$$

where for $i = 2, 3, 4, \cdots$, $\bar{f}_s^i(\bar{x}_c, \bar{x}_s = 0) = 0$.

In this section, a general method for the construction of Lyapunov functions for systems of the form (5.3), (5.4) which are known to be LAS will be provided. This construction depends on Lyapunov functions $\overline{V}_c(\bar{x}_c)$ and $\overline{V}_s(\bar{x}_s)$ associated with the critical and stable dynamics, respectively. The following lemma shows the stable dynamics (5.4) are always LAS for $\|\bar{x}_c\|$ small enough.

Lemma 5 *If $\|\bar{x}_c\|$ is small enough, the dynamics of the stable states (5.4) are LAS.* ◇◇◇

Proof: Since $\overline{f}_c^i(\bar{x}_c, 0) = 0$ for $i = 2, 3, 4, \cdots$ in (5.4), the dynamics for $\bar{x}_s$ can be rewritten as

$$\dot{\bar{x}}_s = A^s \bar{x}_s + G(\bar{x}_c, \bar{x}_s)\bar{x}_s \tag{5.5}$$

where $G(\overline{x}_c, \overline{x}_s)$ is an $n_s \times n_s$ matrix whose entries are continuous functions of the components of $\overline{x}_c$ and $\overline{x}_s$, with $G(0,0) = 0$. Let $P = P^T > 0$ be a real matrix such that $(A^s)^T P + PA^s < 0$. Taking $\overline{V}_s(\overline{x}_s) = \overline{x}_s^T P \overline{x}_s$ as a Lyapunov function candidate for (5.4) yields

$$\overline{W}_s(\overline{x}_c, \overline{x}_s) \stackrel{def}{=} \dot{\overline{V}}_s(\overline{x}_s) = \overline{x}_s^T \{(A^s)^T P + PA^s\} \overline{x}_s + \overline{x}_s^T \{G^T(\overline{x}_c, \overline{x}_s) P + PG(\overline{x}_c, \overline{x}_s)\} \overline{x}_s. \tag{5.6}$$

There exists some neighborhood of $\overline{x}_s = 0$, such that in this neighborhood, each entry of the matrix $G^T(\overline{x}_c, \overline{x}_s)P + PG(\overline{x}_c, \overline{x}_s)$ will be arbitrarily small, if $||\overline{x}_c||$ is small enough. Thus, $G^T(\overline{x}_c, \overline{x}_s)P + PG(\overline{x}_c, \overline{x}_s)$ can be treated as a small perturbation to the matrix $(A^s)^T P + PA^s$, which remains negative definite under sufficiently small perturbations. Therefore, for $||\overline{x}_c||$ small enough, $\overline{W}_s(\overline{x}_c, 0) = 0$ and $\overline{W}_s(\overline{x}_c, \overline{x}_s) < 0$ for $\overline{x}_s \neq 0$ in some neighborhood of $\overline{x}_s = 0$. Thus, the subsystem (5.4) is LAS for $||\overline{x}_c||$ small enough. Furthermore, $\overline{V}_s(\overline{x}_s) = \overline{x}_s^T P \overline{x}_s$ is a Lyapunov function of the subsystem (5.4). ◊◊◊

Since the dynamics of the critical states (5.3) are decoupled from those of the stable states (5.4), and the dynamics of the stable states are always LAS if $||\overline{x}_c||$ is small enough, the LASY of the entire system (5.3), (5.4) (or equivalently, (5.1), (5.2)) is determined by the LASY of the subsystem (5.3). A Lyapunov function for the entire system (5.3), (5.4 can be constructed if a Lyapunov function which determines the LAS of (5.3) is available. This is summarized as follows:

Theorem 21 *When a full normal form (5.3), (5.4) of a system described by (5.1) and (5.2) exists and the corresponding transformation is given by $\bar{x} = v(x)$, the LASY of the system (5.1), (5.2), (or equivalently, (5.3), (5.4)), is completely characterized by that of (5.3). If*

the subsystem (5.3) is LAS, and $\overline{V}_c(\overline{x}_c)$ is a Lyapunov function for the dynamics (5.3), then a Lyapunov function for the entire system (5.3), (5.4) is given by $\overline{V}(\overline{x}_c, \overline{x}_s) = \overline{V}_s(\overline{x}_s) + \overline{V}_c(\overline{x}_c)$, where $\overline{V}_s(\overline{x}_s)$ is constructed as in Lemma 5, (or is any other Lyapunov function for (5.4)). A Lyapunov function for (5.1)-(5.2) is given by $V(x_c, x_s) = \overline{V}(\overline{x}_c, \overline{x}_s)|_{\bar{x}=v(x)}$ $\diamond\diamond\diamond$

Proof: By construction, $\overline{V}(\overline{x}_c, \overline{x}_s)$ is locally positive definite. Consider

$$\dot{\overline{W}}(\overline{x}_c, \overline{x}_s) \stackrel{def}{=} \dot{\overline{V}}(\overline{x}_c, \overline{x}_s) = \dot{\overline{V}}_c(\overline{x}_c) + \dot{\overline{V}}_s(\overline{x}_s) \stackrel{def}{=} \overline{W}_c(\overline{x}_c) + \overline{W}_s(\overline{x}_c, \overline{x}_s).$$

By hypothesis, in some neighborhood of $\overline{x}_s = 0$; $\overline{x}_c = 0$,

$$\overline{W}(0,0) = 0$$

$$\overline{W}(\overline{x}_c, 0) = \overline{W}_c(\overline{x}_c) + \overline{W}_s(\overline{x}_c, 0) = \overline{W}_c(\overline{x}_c) < 0; \quad \overline{x}_c \neq 0$$

$$\overline{W}(0, \overline{x}_s) = \overline{W}_s(0, \overline{x}_s) < 0; \quad \overline{x}_s^T \neq 0$$

$$\overline{W}(\overline{x}_c, \overline{x}_s) = \overline{W}_c(\overline{x}_c) + \overline{W}_s(\overline{x}_c, \overline{x}_s) < 0; \quad \overline{x}_c \neq 0, \overline{x}_s \neq 0.$$

Therefore, $\overline{V}(\overline{x}_c, \overline{x}_s) = \overline{V}_s(\overline{x}_s) + \overline{V}_c(\overline{x}_c)$ is a Lyapunov function with a locally negative definite derivative along the dynamics of the entire system (5.3), (5.4), so that (5.3), (5.4) is LAS.

Now taking $V(x_c, x_s) = \bar{V}(\overline{x}_c, \overline{x}_s)|_{\bar{x}=v(x)}$ as a Lyapunov function candidate for the system (5.1) and (5.2) yields

$$\dot{W}(x_c, x_s) \stackrel{def}{=} \frac{dV(x_c, x_s)}{dt} = \frac{d\overline{V}(\overline{x}_c, \overline{x}_s)}{dt}|_{\bar{x}=v(x)} = \dot{\overline{W}}(\overline{x}_c, \overline{x}_s)|_{\bar{x}=v(x)}.$$

The local positive (or negative) definiteness of the function $V(x_c, x_s)$ (or $W(x_c, x_s)$) follows since the transformation $\bar{x} = v(x)$ is a local bijection (since the Jacobian matrix at the origin

is identity). hence the function $V(x_c, x_s)$ is a Lyapunov function for the system (5.1) and (5.2) ◊◊◊

Although Theorem 21 gives a general method for constructing a Lyapunov function for an LAS nonlinear system of the form (5.1), (5.2), it is not very practical. First of all, the existence of the full normal form is not always guaranteed. Secondly, even if a full normal form does exist, the derivation of this full normal form requires one to solve a partial differential equation. This is impossible in general. However, for any critical nonlinear system, an approximate normal form (say up to degree k) can always be obtained. Thus, it is desirable to use approximate normal forms for construction of Lyapunov functions. As in Theorem 21, the construction of a Lyapunov function for an LAS critical nonlinear system which is in an approximate normal form will depend on the availability of a Lyapunov function for the associated purely critical nonlinear subsystem. A general method for constructing Lyapunov functions for purely critical nonlinear systems is known only for two special cases: either the purely critical system has one zero eigenvalue, or it has a pair of purely imaginary eigenvalues. For LAS critical nonlinear systems with such critical dynamics, Lyapunov functions can be constructed by only using approximate normal forms rather than full normal forms, as will be shown in next section.

5.2 Construction of Lyapunov Functions for Nonlinear Systems in Two Special Cases

In this section, the stability or instability of nonlinear systems in two special cases is studied using Lyapunov functions and approximate normal forms. These two cases are:

CASE 1: Nonlinear systems of the form (5.1) and (5.2) with $A^c = 0$ and $x_c \in R^1$.

CASE 2: Nonlinear systems of the form (5.1) and (5.2) with $A^c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $x_c \in R^2$.

There are many stability or instability theorems derived from Lyapunov's theory, [37], [42]. A variation of the instability theorem which is used intensively in rest of the dissertation will be stated here.

Theorem 22 [37] (Instability Theorem)

Consider a nonlinear system $\dot{x} = f(x)$. Let $x = 0$ be an equilibrium point for this system. Let $V : D \to R$ be a continuously differentiable function on a neighborhood D of $x = 0$, such that $V(0) = 0$ and $V(x)$ is locally indefinite. The equilibrium point $x = 0$ is unstable if $\dot{V}(x)$ is locally definite (either positive or negative definite). ◇◇◇

5.2.1 CASE 1: Nonlinear Systems with One Zero Eigenvalue

Consider a nonlinear system whose linearization possesses only one zero eigenvalue with all others being stable. An approximate normal form (up to order k), where $a_k \neq 0$ is the first nonzero real number in the sequence $\{a_2, a_3, a_4, \cdots\}$, has the following structure:

$$\dot{\overline{x}}_c = a_k \overline{x}_c^k + \alpha(|\overline{x}_c, \overline{x}_s|^{k+1}), \tag{5.7}$$

$$\begin{aligned} \dot{\overline{x}}_s &= A^s \overline{x}_s + \overline{f}_s^2(\overline{x}_c, \overline{x}_s) + \cdots + \overline{f}_s^k(\overline{x}_c, \overline{x}_s) + \beta(|\overline{x}_c, \overline{x}_s|^{k+1}) \\ &= A^s \overline{x}_s + G(\overline{x}_c, \overline{x}_s)\overline{x}_s + \beta(|\overline{x}_c, \overline{x}_s|^{k+1}) \ , \end{aligned} \tag{5.8}$$

where $G(\overline{x}_c, \overline{x}_s)$ is an $n_s \times n_s$ matrix function in the components of $\overline{x}_c$ and $\overline{x}_s$ with $G(\overline{x}_c, \overline{x}_s)\overline{x}_s = \overline{f}_s^2(\overline{x}_c, \overline{x}_s) + \cdots + \overline{f}_s^k(\overline{x}_c, \overline{x}_s)$, and $G(0,0) = 0$. This factorization is guaranteed since $\overline{f}_s^i(\overline{x}_c, \overline{x}_s)$, $i = 2, 3, \cdots, k$ satisfies $\overline{f}_s^i(\overline{x}_c, 0) = 0$. The symbols $\alpha(|\overline{x}_c, \overline{x}_s|^{k+1})$ and $\beta(|\overline{x}_c, \overline{x}_s|^{k+1})$ are used to denote those terms of degree $k+1$ or higher.

Theorem 23 The system (5.7)-(5.8) is LAS if k is odd and $a_k < 0$. If $a_k \neq 0$ when k is even, or $a_k > 0$ when k is odd, the system is unstable. ◇◇◇

Proof:

- Assume that k is odd and $a_k < 0$. Let $\overline{V}_s(\overline{x}_s) = \overline{x}_s^T P \overline{x}_s$, and $\overline{V}_c(\overline{x}_c) = \frac{1}{2}\overline{x}_c^2$, where the real matrix $P = P^T > 0$ is such that $(A^s)^T P + P A^s = Q < 0$. Let

$$\overline{V}(\overline{x}_c, \overline{x}_s) = \overline{V}_c(\overline{x}_c) + \overline{V}_s(\overline{x}_s) = \frac{1}{2}\overline{x}_c^2 + \overline{x}_s^T P \overline{x}_s.$$

The time derivative of $\overline{V}(\overline{x}_c, \overline{x}_s)$ along the dynamics (5.7), (5.8) is

$$\begin{aligned} \dot{\overline{V}}(\overline{x}_c, \overline{x}_s) &= a_k \overline{x}_c^{k+1} + \overline{x}_s\{Q + H(\bar{x})\}\overline{x}_s \\ &\quad + \overline{x}_c\, \alpha(|\overline{x}_c, \overline{x}_s|^{k+1}) + 2\,\overline{x}_s^T P\, \beta(|\overline{x}_c, \overline{x}_s|^{k+1}). \end{aligned} \tag{5.9}$$

where $H(\bar{x}) \stackrel{def}{=} G(\overline{x}_c, \overline{x}_s)^T P + P G(\overline{x}_c, \overline{x}_s)$. The terms of degree $k+1$ or higher can be grouped as follows:

$$\overline{x}_c\, \alpha(|\overline{x}_c, \overline{x}_s|^{k+1}) + 2\,\overline{x}_s^T P\, \beta(|\overline{x}_c, \overline{x}_s|^{k+1}) = \overline{x}_c^{k+1} \Phi(\overline{x}_c, \overline{x}_s) + \overline{x}_s^T \Lambda(\overline{x}_c, \overline{x}_s)\overline{x}_s$$

where $\Phi(\overline{x}_c, \overline{x}_s)$, and $\Lambda(\overline{x}_c, \overline{x}_s) = \Lambda^T(\overline{x}_c, \overline{x}_s)$ are some functions of $(\overline{x}_c, \overline{x}_s)$ with $\Phi(0,0) = 0$, and $\Lambda(0,0) = 0$. The local negative definiteness of (5.9) follows from the equation below:

$$\dot{\overline{V}}(\overline{x}_c, \overline{x}_s) = \overline{x}_c^{k+1}(a_k + \Phi(\overline{x}_c, \overline{x}_s)) + \bar{x}_s^T \{Q + H(\bar{x}) + \Lambda(\bar{x})\}\, \bar{x}_s.$$

- Assume that k is odd, and $a_k > 0$. It is clear that $\overline{V}(\overline{x}_c, \overline{x}_s) = -\frac{1}{2}\overline{x}_c^2 + \overline{x}_s^T P \overline{x}_s$ is locally indefinite, and

$$\dot{\overline{V}}(\overline{x}_c, \overline{x}_s) = -a_k \overline{x}_c^{k+1} + \overline{x}_s\{(Q + H(\bar{x})\}\overline{x}_s$$

$$-\overline{x}_c\,\alpha(|\overline{x}_c,\overline{x}_s|^{k+1}) + 2\,\overline{x}_s^T\,P\,\beta(|\overline{x}_c,\overline{x}_s|^{k+1}), \qquad (5.10)$$

is locally negative definite. The system is unstable when $a_k > 0$, for k odd, by the instability theorem.

- Assume that k is even and $a_k \neq 0$. It is clear that $\overline{V}(\overline{x}_c,\overline{x}_s) = -a_k\overline{x}_c + \overline{x}_s^T P\overline{x}_s$ is locally indefinite, and

$$\begin{aligned}\dot{\overline{V}}\,(\overline{x}_c,\overline{x}_s) &= -a_k^2\,\overline{x}_c^k + \overline{x}_s\{(Q + H(\bar{x})\}\overline{x}_s \\ &\quad -a_k\,\alpha(|\overline{x}_c,\overline{x}_s|^{k+1}) + 2\,\overline{x}_s^T\,P\,\beta(|\overline{x}_c,\overline{x}_s|^{k+1}), \qquad (5.11)\end{aligned}$$

with terms of degree $k+1$ or higher be grouped as

$$-a_k\,\alpha(|\overline{x}_c,\overline{x}_s|^{k+1}) + 2\,\overline{x}_s^T\,P\,\beta(|\overline{x}_c,\overline{x}_s|^{k+1}) = \overline{x}_c^k\Phi(\overline{x}_c,\overline{x}_s) + \overline{x}_s^T\Lambda(\overline{x}_c,\overline{x}_s)\overline{x}_s\,,$$

for some functions $\Phi(\overline{x}_c,\overline{x}_s)$, and $\Lambda(\overline{x}_c,\overline{x}_s) = \Lambda^T(\overline{x}_c,\overline{x}_s)$, with $\Phi(0,0) = 0$, and $\Lambda(0,0) = 0$. The local negative definiteness of (5.11) follows directly from this expression, and the system is hence unstable for k even.

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5.2.2 CASE 2: Nonlinear Systems with a Pair of Purely Imaginary Eigenvalues

Consider a nonlinear system whose linearization possesses a pair purely imaginary with all other eigenvalues being stable. An approximate normal form (up to order $(2k+1)$), where $m_{(2k+1)} \neq 0$ is the first nonzero real number in the sequence $\{m_3, m_5, m_7, \cdots\}$, has

the following structure:

$$\dot{\overline{x}}_{c1} = -\omega\overline{x}_{c2} - (\overline{x}_{c1}^2 + \overline{x}_{c2}^2)n_3\overline{x}_{c2} + \cdots - (\overline{x}_{c1}^2 + \overline{x}_{c2}^2)^{(k-1)}n_{(2k-1)}\overline{x}_{c2}$$
$$-(\overline{x}_{c1}^2 + \overline{x}_{c2}^2)^k(n_{(2k+1)}\overline{x}_{c2} - m_{(2k+1)}\overline{x}_{c1}) + O(|\overline{x}_c, \overline{x}_{c2}|^{2k+2}) \quad (5.12)$$

$$\dot{\overline{x}}_{c2} = \omega\overline{x}_{c1} + (\overline{x}_{c1}^2 + \overline{x}_{c2}^2)n_3\overline{x}_{c1} + \cdots + (\overline{x}_{c1}^2 + \overline{x}_{c2}^2)^{(k-1)}n_{(2k-1)}\overline{x}_{c1}$$
$$+(\overline{x}_{c1}^2 + \overline{x}_{c2}^2)^k(m_{(2k+1)}\overline{x}_{c2} + n_{(2k+1)}\overline{x}_{c1}) + O(|\overline{x}_c, \overline{x}_s|^{(2k+2)}), \quad (5.13)$$

$$\dot{\overline{x}}_s = A^s\overline{x}_s + \overline{f}_s^2(\overline{x}_c, \overline{x}_s) + \cdots + \overline{f}_s^{(2k+1)}(\overline{x}_c, \overline{x}_s) + O(|\overline{x}_c, \overline{x}_s|^{(2k+2)})$$
$$= A^s\overline{x}_s + G(\overline{x}_c, \overline{x}_s)\overline{x}_s + O(|\overline{x}_c, \overline{x}_s|^{2k+2}) \ , \quad (5.14)$$

where $m_{(2k+1)} \neq 0$, and $G(\overline{x}_c, \overline{x}_s)$ is an $n_s \times n_s$ matrix function in the components of $\overline{x}_s$ and $\overline{x}_c$, with $G(\overline{x}_c, \overline{x}_s)\overline{x}_s = \overline{f}_s^2(\overline{x}_c, \overline{x}_s) + \cdots + \overline{f}_s^k(\overline{x}_c, \overline{x}_s)$, and $G(0,0) = 0$. The representation (5.14) is due to the fact that $\overline{f}_s^i(\overline{x}_c, 0) = 0$, $i = 2, 3, \cdots, k$. The terms $O(|\overline{x}_c, \overline{x}_s|^{2k+2})$ denote terms of degree $2k+2$ or higher.

Theorem 24 The system (5.12), (5.13), (5.14) is LAS if and only if $m_{2k+1} < 0$. ◇◇◇

Proof of Theorem 24: Using polar coordinates

$$\overline{x}_{c2} = r\cos\theta$$

$$\overline{x}_{c1} = r\sin\theta$$

The dynamics (5.12), (5.13), (5.14) is transformed to

$$\dot{r} = m_{2k+1}r^{2k+1} + O(|r, \overline{x}_{c2}|^{2k+2}) \quad (5.15)$$

$$\dot{\overline{x}}_s = A^s\overline{x}_s + G(r, \overline{x}_s)\overline{x}_s + O(|r, \overline{x}_s|^{2k+2}), \quad (5.16)$$

where the dynamics for θ have been omitted to simplify the exposition.

The rest of the proof is similar to that of Theorem 23, using $\overline{V}(r,\overline{x}_s) = \frac{1}{2}r^2 + \overline{x}_s^T P \overline{x}_s$ for $m_{2k+1} < 0$, and $\overline{V}(r,\overline{x}_s) = -\frac{1}{2}r^2 + \overline{x}_s^T P \overline{x}_s$ for $m_{2k+1} > 0$. When $m_{2k+1} < 0$, the system (5.12), (5.13) and (5.14) is LAS and $\overline{V}(\overline{x}_c,\overline{x}_s) = \frac{1}{2}(\overline{x}_{c1}^2 + \overline{x}_{c2}^2) + \overline{x}_s^T P \overline{x}_s$ is a Lyapunov function. ◊◊◊

Example 15 (A CASE 1 system)

A Lyapunov function constructed in [24] for the study of the LASY of the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_1^3 \\ \dot{x}_2 &= -x_2 - 2x_1^2 \end{aligned}$$

is given by

$$V(x_1,x_2) = x_1^2 + \frac{1}{16}x_2^2 + \frac{7}{4}x_1^2 x_2.$$

A normal form up to degree 3 is given by

$$\begin{aligned} \dot{\bar{x}}_1 &= -\bar{x}_1^3 + O(|\cdot|^4) \\ \dot{\bar{x}}_2 &= -\bar{x}_2 + 4\,\bar{x}_1^2 \bar{x}_2 + O(|\cdot|^4). \end{aligned}$$

Using Theorem 23, a Lyapunov function for the system in the normal form is given by

$$\overline{V}(\overline{x}_c,\overline{x}_s) = \overline{x}_c^2 + \overline{x}_s^2.$$

5.3 Lyapunov Functions in the Original Coordinates

Once a given critical nonlinear system whose critical dynamics satisfy CASE 1 or CASE 2 is transformed into an appropriate normal form, the stability (or instability) can be easily determined. Furthermore, a Lyapunov function in the normal form coordinates $(\bar{x}_c, \bar{x}_s)$ can be easily constructed if the system is LAS. This Lyapunov function can be used for estimation of some system performance characteristics such as stability region, convergence rate, and robustness, as well as for other applications [24]. However, it is often preferred to perform the stability and performance analysis on the original system in its original coordinates. A Lyapunov function in the original coordinates is hence desirable.

Suppose a nonlinear system, one of its normal forms, and the associated transformation are given respectively as:

$$\begin{aligned}
\dot{x} &= f(x) = \sum_{m=0}^{\infty} f^{m+1}(x)/m! = Ax + f^2(x) + f^3(x)/2! + \cdots \\
\dot{\bar{x}} &= \bar{f}(\bar{x}) = \sum_{m=0}^{\infty} \bar{f}^{m+1}(\bar{x})/m! = A\bar{x} + \bar{f}^2(\bar{x}) + \bar{f}^3(\bar{x})/2! + \cdots \\
x &= u(\bar{x}) = \sum_{m=0}^{\infty} v^{m+1}(\bar{x})/m! = \bar{x} + u^2(\bar{x}) + u^3(\bar{x})/2! + \cdots.
\end{aligned}$$

From the computation algorithm of normal forms and the associated transformations in chapter 3, $\bar{f}^k$ only depends on $f^1(x) = Ax$, f^2, f^3, $\cdots$, f^k and $u^2, u^3, \cdots, u^k$. The choice of $u^{i+1}, i \geq k$ does not effect f^2, f^3, $\cdots$, and f^k. To derive an approximate normal form (say up to degree k), a transformation can be chosen as $x = u(\bar{x}) = \bar{x} + u^2(\bar{x}) + u^3(\bar{x})/2! + \cdots + u^k(\bar{x})/(k-1)!$. To derive a Lyapunov function in the coordinates x, the inverse transformation $\bar{x} = v(x) = x + v^2(x) + v^3(x)/2! + \cdots$ is needed. In general, the inverse transformation $\bar{x} = v(x)$ does not have a closed form even $x = u(\bar{x})$ does. In what

follows, we will investigate the dependency of $\bar{f}^k$ on $f^1(x) = Ax$, f^2, f^3, $\cdots$, on f^k and $v^2, v^3, \cdots$, and we will also derive an efficient algorithm for the computation of $v^2, v^3, \cdots$.

The homogeneous function $v^2, v^3, \cdots$ can also be computed using Theorem 13 in Chapter 3. We first assume a parameterized inverse transformation $\bar{x} = v(x) = x + v^2(x) + v^3(x)/2! + \cdots$. Let

$$\begin{aligned} p(x,a) &= \sum_{m=0}^{\infty} p_m(x)\frac{a^m}{m!} = \sum_{m=0}^{\infty} v^{m+1}(x)\frac{a^m}{m!}, \\ u(\bar{x},a) &= \sum_{m=0}^{\infty} u_m(x)\frac{a^m}{m!} = \sum_{m=0}^{\infty} u^{m+1}(x)\frac{a^m}{m!}. \end{aligned}$$

Then

$$q(\bar{x},a) = p(u(\bar{x},a),a) = \sum_{m=0}^{\infty} q_m(\bar{x})\frac{a^m}{m!} = \sum_{m=0}^{\infty} q^{m+1}(\bar{x})\frac{a^m}{m!},$$

is computed using the algorithm provided in Theorem 13. By definition, $q(\bar{x},a) = \bar{x}$. Thus, a set of equations

$$q_0(\bar{x}) = \bar{x}, \quad q_i(\bar{x}) = 0, \quad \text{for } i = 1,2,3,\cdots,$$

is obtained. The explicit formulas for computing the first few elements in the sequence $\{v^1, v^2, v^3, v^4, \cdots\}$ are given below:

$$\begin{aligned} q_0(\bar{x}) &= v^1(\bar{x}) = \bar{x} \\ q_1(\bar{x}) &= 0 = v^2(\bar{x}) + U_2(x) \\ q_2(\bar{x}) &= 0 = v^3(\bar{x}) + \frac{\partial v^2(\bar{x})}{\partial \bar{x}}U_2(\bar{x}) + \frac{\partial q_1(\bar{x})}{\partial \bar{x}}U_2(\bar{x}) + U_3(\bar{x}) \\ q_3(\bar{x}) &= 0 = v^4(\bar{x}) + \frac{\partial v^3(\bar{x})}{\partial \bar{x}}U_2 + 2\frac{\partial v^2(\bar{x})}{\partial \bar{x}}U_3 + U_4 \\ &\quad + \frac{\partial}{\partial \bar{x}}\left(v^3(\bar{x}) + \frac{\partial v^2(\bar{x})}{\partial \bar{x}}U_2 + U_3\right)U_2 + \frac{\partial q_1(\bar{x})}{\partial \bar{x}}U_3 + \frac{\partial q_2(\bar{x})}{\partial \bar{x}}U_2. \end{aligned}$$

Each v^m for $m = 1, 2, 3, 4, \cdots$ can be derived by solving these equations. With further manipulations, the solutions of v^m's are given below:

$$
\begin{aligned}
v^1(\bar{x}) = v_0(\bar{x}) &= \bar{x} \\
v^2(y) = v_1(y) &= -U_0(y) \\
v^3(y) = v_2(y) &= -\left(\frac{\partial v_1(y)}{\partial y}U_0(y) + \frac{\partial v_0(y)}{\partial y}U_1(y)\right) \\
v^4(y) = v_3(y) &= -\left(\frac{\partial v_2(y)}{\partial y}U_0(y) + 2\frac{\partial v_1(y)}{\partial y}U_1(y) + \frac{\partial v_0(y)}{\partial y}U_2(y)\right) \\
v^5(y) = v_4(y) &= -\left(\frac{\partial v_3(y)}{\partial y}U_0(y) + 3\frac{\partial v_2(y)}{\partial y}U_1(y) + 3\frac{\partial v_1(y)}{\partial y}U_2(y) + \frac{\partial v_0(y)}{\partial y}U_3(y)\right) \\
v^6(y) = v_5(y) &= -\left(\frac{\partial v_4(y)}{\partial y}U_0(y) + 4\frac{\partial v_3(y)}{\partial y}U_1(y) + 6\frac{\partial v_2(y)}{\partial y}U_2(y)\right. \\
&\quad \left. + 4\frac{\partial v_1(y)}{\partial y}U_3(y) + \frac{\partial v_0(y)}{\partial y}U_4(y)\right).
\end{aligned}
$$

Surprising as it is, the computation of the sequence $\{v^m\}$ is simpler than that of the sequence $\{u^m\}$ when the sequence $\{U_0, U_1, U_2, \cdots\}$ is known.

In general, we have

$$
v^{m+2}(\bar{x}) = v_{m+1}(\bar{x}) = -\sum_{i=0}^{m} \binom{m}{i} \frac{\partial v_{m-i}(y)}{\partial y} U_i(y). \tag{5.17}
$$

Recall that $v_0(\bar{x}) = \bar{x}$. Therefore, for a given sequence $\{U_m\}$, the sequence $\{v_m\}$ is uniquely determined, vice-versa. The following theorem is obtained by observing equation (5.17).

Theorem 25 *Let the original system, one of its normal forms, and the associated transformation be given respectively by:*

$$
\dot{x} = f(x) = \sum_{m=0}^{\infty} f^{m+1}(x)/m! = Ax + f^2(x) + f^3(x)/2! + \cdots \tag{5.18}
$$

$$\dot{\bar{x}} = \bar{f}(\bar{x}) = \sum_{m=0}^{\infty} \bar{f}^{m+1}(\bar{x})/m! = A\bar{x} + \bar{f}^2(\bar{x}) + \bar{f}^3(\bar{x})/2! + \cdots \tag{5.19}$$

$$\bar{x} = v(x) = \sum_{m=0}^{\infty} v^{m+1}(x)/m! = x + v^2(x) + v^3(x)/2! + \cdots. \tag{5.20}$$

Then $\bar{f}^k$ only depends on $f^1(x) = Ax$, f^2, f^3, $\cdots$, and f^k, and $v^2, v^3, \cdots$, and v^k. $\Diamond\Diamond\Diamond$

Finally, we have the following theorem on the construction of Lyapunov functions in the original coordinates if a given nonlinear system is LAS.

Theorem 26 *Given the system*

$$\dot{x} = f(x) = Ax + f^2(x) + f^3(x) + \cdots, \tag{5.21}$$

its normal form, (up to degree k)

$$\dot{\bar{x}} = A\bar{x} + \bar{f}^2(\bar{x}) + \cdots + \bar{f}^k(\bar{x}) + \cdots \tag{5.22}$$

is derived using the transformation

$$\bar{x} = v(x) = x + v^2(x) + \cdots + v^k(x). \tag{5.23}$$

If $\bar{V}(\bar{x})$ is a Lyapunov function for the LASY of (5.22), then

$$V(x) = \bar{V}(\bar{x})|_{\bar{x}=v(x)} \tag{5.24}$$

is a Lyapunov function for the LASY of (5.21). $\Diamond\Diamond\Diamond$

5.4 Examples and Discussions

Several additional examples which are calculated with the help of *Maple* are presented in this section. All of the computations only proceed to degree 3.

Example 16 (A CASE 1 system)

Consider the system:

$$\begin{aligned} \dot{x}_1 &= -x_1 x_2 + x_1 {x_3}^2 + x_1 x_2 x_3, \\ \dot{x}_2 &= -x_2 + {x_1}^2 + x_1 {x_3}^2, \\ \dot{x}_3 &= -x_3 + x_1 x_2 \,. \end{aligned} \tag{5.25}$$

A normal form is given by

$$\begin{aligned} \dot{\bar{x}}_1 &= -\bar{x}_1^3 + O(|\cdot|^4), \\ \dot{\bar{x}}_2 &= -\bar{x}_2 + 2\,\bar{x}_1^2 \bar{x}_2 + O(|\cdot|^4), \\ \dot{\bar{x}}_3 &= -\bar{x}_3 + \bar{x}_1 \bar{x}_2 + O(|\cdot|^4). \end{aligned} \tag{5.26}$$

The associated backward transformation is

$$\begin{aligned} x_1 &= \bar{x}_1 + \bar{x}_1 \bar{x}_2 - \frac{\bar{x}_1 \bar{x}_2 \bar{x}_3}{2} - \frac{\bar{x}_1 \bar{x}_3^2}{2} + \frac{\bar{x}_1 \bar{x}_2^2}{2} + \frac{\bar{x}_1^3}{2}, \\ x_2 &= \bar{x}_2 + \bar{x}_1^2 - \bar{x}_1 \bar{x}_3^2 + \bar{x}_1^2 \bar{x}_2, \\ x_3 &= \bar{x}_3 + \bar{x}_1^3 - \bar{x}_1 \bar{x}_2^2. \end{aligned} \tag{5.27}$$

The associated forward transformation is

$$\begin{aligned}
\bar{x}_1 &= x_1 - x_1x_2 + \frac{{x_1}^3}{2} + \frac{x_1{x_2}^2}{2} + \frac{x_1x_2x_3}{2} + \frac{x_1{x_3}^2}{2}, \\
\bar{x}_2 &= x_2 - {x_1}^2 + {x_1}^2x_2 + x_1{x_3}^2, \\
\bar{x}_3 &= x_3 - {x_1}^3 + x_1{x_2}^2 .
\end{aligned} \tag{5.28}$$

It is easy to see that the system (5.26) by Theorem 23 is LAS using

$$\bar{V}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2,$$

as a Lyapunov function. A Lyapunov function for the LASY of the system (5.25) is given by

$$\begin{aligned}
V(x_1, x_2, x_3) &= \left(x_1 - x_1x_2 + \frac{x_1^3}{2} + \frac{x_1{x_2}^2}{2} + \frac{x_1x_2x_3}{2} + \frac{x_1{x_3}^2}{2} \right)^2 \\
&\quad + \left(x_2 - {x_1}^2 + {x_1}^2x_2 + x_1{x_3}^2 \right)^2 + \left(x_3 - {x_1}^3 + x_1{x_2}^2 \right)^2 .
\end{aligned}$$

Example 17 (A CASE 2 system)

Consider a Case 2 system:

$$\begin{aligned}
\dot{x}_1 &= -x_2 + x_1{x_3}^2 + x_1x_2x_3 + {x_1}^2, \\
\dot{x}_2 &= x_1 + {x_1}^2 + x_1{x_3}^2, \\
\dot{x}_3 &= -x_3 + x_1x_2 .
\end{aligned} \tag{5.29}$$

A normal form up to degree 3 is given by

$$
\begin{aligned}
\dot{\bar{x}}_1 &= -\bar{x}_2 - \frac{(3\,\bar{x}_1 - 7\,\bar{x}_2)\,(\bar{x}_1^2 + \bar{x}_2^2)}{12} + O(|\cdot|^4)\,, \\
\dot{\bar{x}}_2 &= \bar{x}_1 - \frac{(3\,\bar{x}_2 + 7\,\bar{x}_1)\,(\bar{x}_1^2 + \bar{x}_2^2)}{12} + O(|\cdot|^4)\,, \\
\dot{\bar{x}}_3 &= -\bar{x}_3 + O(|\cdot|^4)\,.
\end{aligned}
\tag{5.30}
$$

The associated backward transformation is given by

$$
\begin{aligned}
x_1 &= \bar{x}_1 - \frac{\bar{x}_1^2}{3} + \frac{2\,\bar{x}_1\bar{x}_2}{3} - \frac{2\,\bar{x}_2^2}{3} + \frac{7\,\bar{x}_1^3}{18} - \frac{\bar{x}_1^2\bar{x}_2}{2} - \frac{\bar{x}_1^2\bar{x}_3}{5} \\
&\quad + \frac{29\,\bar{x}_1\bar{x}_2^2}{36} - \frac{3\,\bar{x}_1\bar{x}_2\bar{x}_3}{10} - \frac{\bar{x}_1\bar{x}_3^2}{2} - \frac{11\,\bar{x}_2^3}{12} + \frac{\bar{x}_2^2\bar{x}_3}{5} + \frac{\bar{x}_2\bar{x}_3^2}{4}\,, \\
x_2 &= \bar{x}_2 + \frac{\bar{x}_1^2}{3} + \frac{2\,\bar{x}_1\bar{x}_2}{3} + \frac{2\,\bar{x}_2^2}{3} + \frac{\bar{x}_1^3}{12} + \frac{11\,\bar{x}_1^2\bar{x}_2}{36} + \frac{\bar{x}_1^2\bar{x}_3}{10} \\
&\quad - \frac{\bar{x}_1\bar{x}_2\bar{x}_3}{10} - \frac{\bar{x}_1\bar{x}_3^2}{4} - \frac{\bar{x}_2^2\bar{x}_3}{10} + \frac{2\,\bar{x}_2^3}{9} + \frac{2\,\bar{x}_1\bar{x}_2^2}{3}\,, \\
x_3 &= \bar{x}_3 - \frac{\bar{x}_1^2}{5} + \frac{\bar{x}_1\bar{x}_2}{5} + \frac{\bar{x}_2^2}{5} + \frac{2\,\bar{x}_1^3}{5} - \frac{\bar{x}_1^2\bar{x}_2}{15} + \frac{4\,\bar{x}_1\bar{x}_2^2}{5} + \frac{2\,\bar{x}_2^3}{15}\,.
\end{aligned}
\tag{5.31}
$$

The associated forward transformation is given by

$$
\begin{aligned}
\bar{x}_1 &= x_1 + \frac{{x_1}^2}{3} - \frac{2\,x_1x_2}{3} + \frac{2\,{x_2}^2}{3} + \frac{{x_1}^3}{18} - \frac{{x_1}^2x_2}{6} + \frac{{x_1}^2x_3}{5} \\
&\quad - \frac{13\,x_1{x_2}^2}{36} + \frac{3\,x_1x_2x_3}{10} + \frac{x_1{x_3}^2}{2} - \frac{5\,{x_2}^3}{12} - \frac{{x_2}^2x_3}{5} - \frac{x_2{x_3}^2}{4}\,, \\
\bar{x}_2 &= x_2 - \frac{{x_1}^2}{3} - \frac{2\,x_1x_2}{3} - \frac{2\,{x_2}^2}{3} - \frac{{x_1}^3}{12} + \frac{29\,{x_1}^2x_2}{36} - \frac{{x_1}^2x_3}{10} \\
&\quad + \frac{2\,x_1{x_2}^2}{3} + \frac{x_1x_2x_3}{10} + \frac{x_1{x_3}^2}{4} + \frac{2\,{x_2}^3}{9} + \frac{{x_2}^2x_3}{10}\,, \\
\bar{x}_3 &= x_3 + \frac{{x_1}^2}{5} - \frac{x_1x_2}{5} - \frac{{x_2}^2}{5} - \frac{{x_1}^3}{5}\,.
\end{aligned}
\tag{5.32}
$$

The system (5.30) is LAS by Theorem 24. A Lyapunov function for (5.30) can be chosen as

$$\bar{V}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 .$$

A Lyapunov function for the LASY of the system (5.29) is given by

$$\begin{aligned} V(x_1, x_2, x_3) \;=\; & \left(x_1 + \frac{{x_1}^2}{3} - \frac{2\,x_1 x_2}{3} + \frac{2\,{x_2}^2}{3} + \frac{23\,{x_1}^3}{36} - \frac{5\,{x_1}^2 x_2}{12} + \frac{{x_1}^2 x_3}{5} \right. \\ & \left. + \frac{2\,x_1 {x_2}^2}{9} + \frac{3\,x_1 x_2 x_3}{10} + \frac{x_1 {x_3}^2}{2} - \frac{2\,{x_2}^3}{3} - \frac{{x_2}^2 x_3}{5} - \frac{x_2 {x_3}^2}{4} \right)^2 \\ & + \left(x_2 - \frac{{x_1}^2}{3} - \frac{2\,x_1 x_2}{3} - \frac{2\,{x_2}^2}{3} + \frac{{x_1}^3}{6} + \frac{25\,{x_1}^2 x_2}{18} - \frac{{x_1}^2 x_3}{10} \right. \\ & \left. + \frac{11\,x_1 {x_2}^2}{12} + \frac{x_1 x_2 x_3}{10} + \frac{x_1 {x_3}^2}{4} + \frac{29\,{x_2}^3}{36} + \frac{{x_2}^2 x_3}{10} \right)^2 \\ & + \left(x_3 + \frac{{x_1}^2}{5} - \frac{x_1 x_2}{5} - \frac{{x_2}^2}{5} - \frac{{x_1}^3}{5} \right)^2 . \end{aligned} \tag{5.33}$$

The expression for $\dot{V}(x_1, x_2, x_3)$ can also be written out explicitly. This expression is intentionally omitted due to its complexity. The local negative definiteness of $\dot{V}(x_1, x_2, x_3)$ is guaranteed by Theorem 24. Since the Hessian of $\dot{V}(x_1, x_2, x_3)$ at the origin is negative semi-definite, the local negative definiteness of $\dot{V}(x_1, x_2, x_3)$ is quite difficult to verify using other methods.

Example 18 (A CASE 2 system)

Consider a simple two-dimensional Case 2 system:

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1^2 \\ \dot{x}_2 &= x_1 + x_1^2 . \end{aligned} \tag{5.34}$$

A normal form up to degree 3 is

$$\begin{aligned}
\dot{\bar{x}}_1 &= -\bar{x}_2 - \frac{\bar{x}_1^3}{4} + \frac{7\,\bar{x}_1^2\bar{x}_2}{12} - \frac{\bar{x}_1\bar{x}_2^2}{4} + \frac{7\,\bar{x}_2^3}{12} + O(|\cdot|^4) \\
&= -\bar{x}_2 - \frac{(3\,\bar{x}_1 - 7\,\bar{x}_2)\,(\bar{x}_1^2 + \bar{x}_2^2)}{12} + O(|\cdot|^4) \\
\dot{\bar{x}}_2 &= \bar{x}_1 - \frac{7\,\bar{x}_1^3}{12} - \frac{\bar{x}_1^2\bar{x}_2}{4} - \frac{7\,\bar{x}_1\bar{x}_2^2}{12} - \frac{\bar{x}_2^3}{4} + O(|\cdot|^4) \\
&= \bar{x}_1 - \frac{(3\,\bar{x}_2 + 7\,\bar{x}_1)\,(\bar{x}_1^2 + \bar{x}_2^2)}{12} + O(|\cdot|^4)\,. \qquad (5.35)
\end{aligned}$$

The associated backward form transformation is given by

$$\begin{aligned}
x_1 &= \bar{x}_1 - \frac{\bar{x}_1^2}{3} + \frac{2\,\bar{x}_1\bar{x}_2}{3} - \frac{2\,\bar{x}_2^2}{3} + \frac{23\,\bar{x}_1\bar{x}_2^2}{36} - \frac{3\,\bar{x}_2^3}{4} + \frac{2\,\bar{x}_1^3}{9} - \frac{\bar{x}_1^2\bar{x}_2}{3} \\
x_2 &= \bar{x}_2 + \frac{\bar{x}_1^2}{3} + \frac{2\,\bar{x}_1\bar{x}_2}{3} + \frac{2\,\bar{x}_2^2}{3} - \frac{\bar{x}_1^3}{12} + \frac{5\,\bar{x}_1^2\bar{x}_2}{36} + \frac{\bar{x}_1\bar{x}_2^2}{2} + \frac{\bar{x}_2^3}{18}\,. \qquad (5.36)
\end{aligned}$$

The associated forward transformation is given by

$$\begin{aligned}
\bar{x}_1 &= x_1 + \frac{{x_1}^2}{3} - \frac{2\,x_1x_2}{3} + \frac{2\,{x_2}^2}{3} + \frac{2\,{x_1}^3}{9} - \frac{{x_1}^2x_2}{3} - \frac{7\,x_1{x_2}^2}{36} - \frac{7\,{x_2}^3}{12} \\
\bar{x}_2 &= x_2 - \frac{{x_1}^2}{3} - \frac{2\,x_1x_2}{3} - \frac{2\,{x_2}^2}{3} + \frac{{x_1}^3}{12} + \frac{35\,{x_1}^2x_2}{36} + \frac{5\,x_1{x_2}^2}{6} + \frac{7\,{x_2}^3}{18}\,. \qquad (5.37)
\end{aligned}$$

The LASY of the system (5.37) is evident by Theorem 24. A Lyapunov function is given by

$$\bar{V}(\bar{x}_1, \bar{x}_2) = \bar{x}_1^2 + \bar{x}_2^2\,.$$

A Lyapunov function for the system in the original coordinates, (5.34), is given by

$$\begin{aligned}
V(x_1, x_2) = &\left(x_1 + \frac{{x_1}^2}{3} - \frac{2\,x_1x_2}{3} + \frac{2\,{x_2}^2}{3} + \frac{2\,{x_1}^3}{9}\right. \\
&\left. - \frac{{x_1}^2x_2}{3} - \frac{7\,x_1{x_2}^2}{36} - \frac{7\,{x_2}^3}{12}\right)^2
\end{aligned}$$

$$\left(+x_2 - \frac{{x_1}^2}{3} - \frac{2\,x_1 x_2}{3} - \frac{2\,{x_2}^2}{3} + \frac{{x_1}^3}{12} + \frac{35\,{x_1}^2 x_2}{36} + \frac{5\,x_1 {x_2}^2}{6} + \frac{7\,{x_2}^3}{18}\right)^2 . \tag{5.38}$$

The local positive definiteness of $V(x_1, x_2)$ is obvious.

The time derivative of $V(x_1, x_2)$ along the dynamics (5.34) is given by

$$\begin{aligned}\frac{dV}{dt} &= \frac{247\,{x_1}^6 x_2}{108} - \frac{173\,{x_1}^6}{216} + \frac{19\,{x_1}^7}{54} - \frac{7\,{x_2}^6}{8} + \frac{10\,{x_2}^5}{9} \\ &+ \frac{32\,{x_1}^3 {x_2}^2}{9} + \frac{61\,{x_1}^2 {x_2}^3}{9} + \frac{413\,{x_2}^5 {x_1}^2}{108} - \frac{212\,{x_1}^3 {x_2}^3}{27} \\ &+ \frac{983\,{x_1}^5 {x_2}^2}{108} - \frac{1703\,{x_1}^2 {x_2}^4}{216} + \frac{1315\,{x_1}^4 {x_2}^3}{108} - \frac{29\,x_1 {x_2}^5}{36} \\ &+ \frac{439\,{x_1}^3 {x_2}^4}{54} - \frac{641\,{x_1}^5 x_2}{108} - \frac{661\,{x_1}^4 {x_2}^2}{72} + \frac{8\,{x_1}^5}{9} \\ &+ 2\,{x_1}^4 x_2 - \frac{{x_1}^4}{2} - \frac{{x_2}^4}{2} - {x_1}^2 {x_2}^2 .\end{aligned} \tag{5.39}$$

Notice that

$$\frac{dV}{dt} = -\frac{1}{2}(x_1^2 + x_2^2)^2 + O(|\cdot|^5) . \tag{5.40}$$

In this case, the local negative definiteness of (5.40) is also evident by using polar coordinates.

This example has been studied by Mees and Chua in [44], as well as by Fu and Abed in [24]. In [24], a Lyapunov function for (5.34) is given as

$$\hat{V}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) - x_1^2 x_2 + \frac{x_1^3}{3} - \frac{2x_2^3}{3} + \frac{x_1 x_2^3}{4} - \frac{x_1^3 x_2}{4} - \frac{x_1^4}{2} . \tag{5.41}$$

The time derivative of $\hat{V}(x_1, x_2)$ along the dynamics (5.34) is given by

$$\frac{d\hat{V}}{dt} = -\frac{1}{4}(x_1^2 + x_2^2)^2 + O(|\cdot|^5). \tag{5.42}$$

It appears that (5.41) is *simpler* than (5.38) in some sense. In what follows, it will be shown that this difference is only superficial. The method proposed in this paper can generate a Lyapunov function which is identical to (5.41), as illustrated below.

As we mentioned before, the normal form for a given system is not unique. It is easily verified that the approximate normal form (5.35) can also be derived using the following transformation

$$\begin{aligned}
\bar{x}_1 &= x_1 + \frac{{x_1}^2}{3} - \frac{2\,x_1 x_2}{3} + \frac{2\,{x_2}^2}{3} + \frac{(1/9 - a_{3,2,4})\,{x_1}^3}{2} \\
&\quad + \frac{(-1/3 + a_{3,2,3})\,{x_1}^2 x_2}{2} + \frac{(-13/18 - a_{3,2,4})}{2} x_1 {x_2}^2 \\
&\quad + \frac{(-5/6 + a_{3,2,3})\,{x_2}^3}{2}\,, \\
\bar{x}_2 &= x_2 - \frac{{x_1}^2}{3} - \frac{2\,x_1 x_2}{3} - \frac{2\,{x_2}^2}{3} + \frac{(-1/6 - a_{3,2,3})\,{x_1}^3}{2} \\
&\quad + \frac{(29/18 - a_{3,2,4})}{2} {x_1}^2 x_2 + \frac{(4/3 - a_{3,2,3})\,x_1 {x_2}^2}{2} \\
&\quad + \frac{(4/9 - a_{3,2,4})\,{x_2}^3}{2}\,,
\end{aligned} \tag{5.43}$$

where $a_{3,2,4}$ and $a_{3,2,3}$ are arbitrary real numbers. The forward transformation (5.37) is obtained by setting $a_{3,2,4} = a_{3,2,3} = -1/3$. Choosing

$$\bar{V}(\bar{x}_1, \bar{x}_2) = \frac{1}{2}\left(\bar{x}_1^2 + \bar{x}_2^2\right),$$

a parameterized Lyapunov function $V_p(x_1, x_2)$ for the system (5.34) derived using the forward transformation (5.43) is obtained by the substitution. Notice that no matter what values $a_{3,2,4}$ and $a_{3,2,3}$ take,

$$\frac{dV_p}{dt} = -\frac{1}{4}(x_1^2 + x_2^2)^2 + O(|\cdot|^5) \,. \tag{5.44}$$

The degree of the lowest order homogeneous polynomial in (5.44) is four, and (5.44) is locally negative definite if its lowest order homogeneous polynomial is negative definite. Therefore, a modified parameterized Lyapunov candidate $V^{new}(x_1, x_2)$, obtained from $V_p(x_1, x_2)$ by truncating terms of degree 5 or higher, is as follows,

$$\begin{aligned} V^{new}(x_1, x_2) \;=\; & \frac{1}{2}\left(x_1^2 + x_2^2\right) - {x_1}^2 x_2 - \frac{2\,{x_2}^3}{3} + \frac{{x_1}^3}{3} - \frac{{x_1}^3 x_2}{4} \\ & + \frac{(-2\,a_{3,2,4} + 8/3)\,{x_1}^2 {x_2}^2}{2} \;+ \frac{x_1 {x_2}^3}{4} \\ & + \frac{(1/3 - a_{3,2,4})\,{x_1}^4}{2} + \frac{(-a_{3,2,4} + 4/3)\,{x_2}^4}{2} \,. \end{aligned}$$

The local positive definiteness of $V^{new}(x_1, x_2)$ is also evident. The Lyapunov function (5.41) is obtained by setting $a_{3,2,4} = 4/3$ in V^{new}. The value of $a_{3,2,3}$ does not matter.

Remark 9 *Actually, all Lyapunov functions constructed in the above examples can be simplified. This topic will be pursued further in the next chapter.* ◇◇◇

5.5 Summary

In this chapter, we first presented a general algorithm for construction of Lyapunov functions for nonlinear critical systems when they are LAS. It was shown that when a given critical nonlinear system is transformed into a full normal form (assume it is existed), a Lyapunov function of the entire system in the normal form can be chosen to be the sum of

the individual Lyapunov functions corresponding to the critical and the stable dynamics, respectively. A Lyapunov function for the system in the original coordinates can be derived by substitution of the appropriate transformation. In the Section 5.2 and 5.3, an explicit algorithm for the construction of Lyapunov functions for two classes of critical nonlinear systems was provided. This construction procedure only required an approximate normal form up to a certain degree. The Section 5.4 provided several examples to highlight use of the algorithm. The issue of simplification of Lyapunov functions is mentioned, and will be pursued further in Chapter 6.

CHAPTER 6
SIMPLIFICATION OF LYAPUNOV FUNCTIONS

In Chapter 5, we proposed an explicit algorithm for the construction of Lyapunov functions of nonlinear systems in either CASE 1 or CASE 2. In this chapter, we will provide a method to refine those constructed Lyapunov functions. As we mentioned before, the construction of Lyapunov functions for nonlinear systems in either CASE 1 or CASE 2 has been studied by some other researchers. A comparison among Lyapunov functions constructed using different methods motivated the work presented in this Chapter. Some heuristic examples are given in Section 6.1. In Section 6.2, we present the method for refinement of Lyapunov functions. Several examples are given in Section 6.3 to illustrate use of the method. This chapter is concluded by a brief summary given in Section 6.4.

6.1 Heuristic Examples

Example 19 (A CASE 1 system)

The system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + x_1^3 \\ \dot{x}_2 &= -x_2 - 2x_1^2 \end{aligned} \tag{6.1}$$

is considered by Fu and Abed in [24]. The origin is an LAS equilibrium point by using center manifold theory. A Lyapunov function constructed in [24] is

$$V(x_1, x_2) = x_1^2 + \frac{1}{16}x_2^2 + \frac{7}{4}x_1^2 x_2.$$

◇◇◇

Furthermore, Fu and Abed proved that if a CASE 1 system is LAS and $a_3 < 0$ is the first nonzero real number is the sequence $\{a_2, a_3, a_4, \cdots\}$, a Lyapunov function including only quadratic and cubic terms can be constructed. Similarly, if a CASE 2 system is LAS and $m_3 < 0$ is the first nonzero real number is the sequence $\{m_3, m_5, m_7, \cdots\}$, a Lyapunov function including only quadratic, cubic and quartic terms can be constructed. The proof of their method is based on the following proposition.

Proposition 1 [24] (Local Definiteness of a Class of Bivariate Functions): The scalar function

$$\begin{aligned}\Lambda(x, y) \;=\; & a_{20}x^2 + a_{30}x^3 + a_{21}x^2 y + a_{12}xy^2 + a_{40}x^4 \\ & + a_{31}x^3 y + a_{22}x^2 y^2 + a_{13}xy^3 + a_{04}y^4 + O(|x, y|^5)\end{aligned}$$

in the real variables x and y is locally negative definite in an open neighborhood of the origin if $a_{20} < 0$, $a_{04} < 0$, and $a_{12}^2 - 4a_{20}a_{04} < 0$. ◇◇◇

If determining the LASY of a nonlinear system needs the consideration of nonlinear terms of degree 4 or higher, Fu and Abed did not give any clue for the construction of Lyapunov functions. On the contrast, the algorithm we proposed in chapter 5 is applicable to any CASE 1 or CASE 2 nonlinear system no matter how complicated the nonlinear

system is. If a nonlinear system is LAS and $a_{2k+1} < 0$ is the first nonzero real number in the sequence $\{a_2, a_3, a_4, \cdots\}$ for CASE 1, (or $m_{2k+1} < 0$ is the first nonzero real number in the sequence $\{m_3, m_5, m_7, \cdots\}$ for CASE 2), Lyapunov functions in the original coordinates (x_c, x_s) which include terms of degrees 2 to $2(2k+1)$ can be constructed constructed using the algorithm provided in Chapter 5 The idea of constructing a simpler Lyapunov function for a given nonlinear system inspired the research presented in this chapter. We want to prove that it is possible to construct Lyapunov functions in the original coordinates (x_c, x_s) that only include terms of degree from 2 to $2k+2$ (instead of $2(2k+1)$).

Example 20 (A CASE 1 system)

In Chapter 5, a Lyapunov function has been constructed for the following CASE 1 system described by

$$\begin{aligned} \dot{x} &= -xy + xz^2 + xyz, \\ \dot{y} &= -y + x^2 + xz^2, \\ \dot{z} &= -z + xy. \end{aligned} \tag{6.2}$$

Choosing a Lyapunov function for the system in the normal form as

$$V(\bar{x}, \bar{y}, \bar{z}) = \bar{x}^2 + \bar{y}^2 + \bar{z}^2,$$

a Lyapunov function for the system in the original coordinates, constructed using the method proposed in Chapter 5, is

$$V(x, y, z) = \left(x - xy + \frac{x^3}{2} + \frac{xy^2}{2} + \frac{xyz}{2} + \frac{xz^2}{2} \right)^2$$

$$+\left(y-x^2+x^2y+xz^2\right)^2+\left(z-x^3+xy^2\right)^2. \tag{6.3}$$

The time derivative along the trajectory,

$$\frac{dV}{dt}=-2\,z^2-2\,y^2+4\,x^2y+2\,xyz+2\,x^3z-4\,x^4-6\,xy^2z-4\,xyz^2+O(|\cdot|^5) \tag{6.4}$$

is guaranteed to be negative definite in at least a small open neighborhood of the origin. Due the complexity of dV/dt here, the terms of degree 5 or higher are omitted. In the following we will prove that a function obtained by truncating the terms of degree 5 or higher from $V(x,y,z)$ in (6.3) is still a Lyapunov function for the system (6.2). First, we need to prove the following Lemma.

Lemma 6 *The function* $\Lambda(x,y,z)=-2\,z^2-2\,y^2+4\,x^2y+2\,xyz+2\,x^3z-4\,x^4-6\,xy^2z-4\,xyz^2$ *is locally negative definite. Any other analytic function* $\tilde{\Lambda}(x,y,z)$ *generated by* $\Lambda(x,y,z)$ *such as* $\tilde{\Lambda}(x,y,z)=\Lambda(x,y,z)+O(|\cdot|^5)$ *is also locally negative definite.* ◇◇◇

Proof: The function $\Lambda(x,y,z)$ can be re-written as follows:

$$\begin{aligned}\Lambda(x,y,z) &= -2\,z^2-2\,y^2+4\,x^2y+2\,xyz+2\,x^3z-4\,x^4-6\,xy^2z-4\,xyz^2\\ &= -a_1y^2+2xyz-a_2z^2 &(6.5)\\ &\quad -a_3y^2+4x^2y-a_4x^4 &(6.6)\\ &\quad -z^2(a_5+4xy) &(6.7)\\ &\quad -y^2(a_6+6xz) &(6.8)\\ &\quad -a_7z^2+2x^3z-a_8x^6 &(6.9)\\ &\quad -x^4(a_9-a_8x^2) &(6.10)\end{aligned}$$

where a_1 - a_9 are some positive real numbers. Recall that $ax^2 + 2bxy + cy^2$ is negative definite if and only if $c < 0$, $b < 0$ and $b^2 - ac < 0$. Therefore, if the conditions

1. $x^2 - a_1 a_2 < 0$,
2. $4 - a_3 a_4 < 0$,
3. $b_1 = a_5 + 4xy > 0$,
4. $b_2 = a_6 + 6xz > 0$,
5. $1 - a_7 a_8 < 0$,
6. $b_3 = a_9 - a_8 x^2 > 0$

hold in a small neighborhood of origin ($x = y = z = 0$) subject to

1. $a_1 + a_3 + a_6 = 2$,
2. $a_2 + a_5 + a_7 = 2$,
3. $a_4 + a_9 = 4$,

$\Lambda(x, y, z)$ will be locally negative definite. These conditions are absolutely achievable providing x, y, and z are small enough. For example, all inequalities hold if we can choose

$$(a_1 = \frac{1}{4}, a_2 = 1, a_3 = \frac{3}{2}, a_4 = \frac{7}{2}, a_5 = \frac{1}{2}, a_6 = \frac{1}{4}, a_7 = \frac{1}{2}, a_8 = 4, a_9 = \frac{1}{2})$$

and $|x|, |y|, |z| < \frac{1}{10}$. The higher order terms, $O(|\cdot|^5)$, can be re-written as

$$O(|\cdot|^5) = y^2 h_1(x, y, x) + yz h_2(x, y, z) + z^2 h_3(x, y, z) + x^4 h_4(x, y, z).$$

where $h_1 - h_4$ are some polynomials vanishing at origin. Consequently, we can re-write $\tilde{\Lambda}(x,y,z)$ as

$$\begin{aligned}\tilde{\Lambda}(x,y,z) &= (-a_1y^2 + 2xyz - a_2z^2) \\ &\quad +(-a_3y^2 + 4x^2y - a_4x^4) \\ &\quad +(-a_7z^2 + 2x^3z - a_8x^6) \\ &\quad -z^2\{(a_5 + 4xy) - h_3(x,y,z)\} + yz\,h_2(x,y,z) \\ &\quad -y^2\{(a_6 + 6xz) - h_1(x,y,z)\} \\ &\quad -x^4\{(a_9 + a_8x^2) - h_4(x,y,z)\}.\end{aligned}$$

The function $\tilde{\Lambda}(x,y,z)$ is indeed locally negative definite. $\diamond\diamond\diamond$

With this lemma a new Lyapunov function, V^{new}, can be derived from (6.3) by truncating the terms of degree 5 or higher. We have

$$\begin{aligned}V^{new}(x,y,z) &= x^2 + y^2 + z^2 - 4\,x^2y + 2\,x^4 - 2\,x^3z + z^2x^2 + 4\,x^2y^2 \\ &\quad +yzx^2 + 2\,xyz^2 + 2\,xy^2z. \qquad (6.11)\end{aligned}$$

One drawback of this approach is that this $V^{new}(x,y,z)$ is only locally positive definite, whereas $V(x,y,z)$ in (6.3) is positive semi-definite globally.

Later we will show that a Lyapunov function for (6.1) which only includes quadratic and cubic terms can also be constructed using the method proposed in the next section.

6.2 Main Results: Simplification of Lyapunov Functions

Due to the similarity between CASE 1 and CASE 2 systems, only CASE 1 systems are considered in this section. Consider a given nonlinear system in CASE 1,

$$\dot{x}_c = A^c x_c + f_c(x_c, x_s), \quad x_c \in R^{n_c}, \tag{6.12}$$

$$\dot{x}_s = A^s x_s + f_s(x_c, x_s), \quad x_s \in R^{n_s}. \tag{6.13}$$

The LASY of the entire system is decided by a set of real numbers $\{a_2, a_3, a_4, a_5, \cdots\}$. This was discussed in Chapter 5. If such a critical system is LAS, then the first nonzero number in sequence $\{a_2, a_3, a_4, a_5, \cdots\}$ must be negative with odd subscript, say $a_{2k+1} < 0$. The normal form up to degree $2k+1$ is given by

$$\dot{\bar{x}}_c = a_{2k+1}\, \bar{x}_c^{2k+1} + O\left(|\bar{x}_c, \bar{x}_s|^{2k+2}\right) \tag{6.14}$$

$$\begin{aligned} \dot{\bar{x}}_s &= A^s \bar{x}_s + \overline{f}_s(\bar{x}_c, \bar{x}_s) \\ &= A^s \bar{x}_s + \overline{f}_s^2(\bar{x}_c, \bar{x}_s) + \cdots + \overline{f}_s^{2k+2}(\bar{x}_c, \bar{x}_s) + O(|\bar{x}_c, \bar{x}_s|^{2k+2}) \\ &= A^s \bar{x}_s + \Theta(\bar{x}_c, \bar{x}_s)\bar{x}_s + O(|\bar{x}_c, \bar{x}_s|^{2k+2}) \end{aligned} \tag{6.15}$$

where $\overline{f}_s^i(\bar{x}_c, \bar{x}_s = 0) = 0$, $i = 1, 2, \cdots, 2k+1$, and $\Theta(\bar{x}_c, \bar{x}_s)\bar{x}_s = \overline{f}_s^2(\bar{x}_c, \bar{x}_s) + \cdots + \overline{f}_s^{2k+1}(\bar{x}_c, \bar{x}_s)$.

Let $P = P^T > 0$ be a matrix such that $A^{sT}P + PA^s < 0$. By Theorem 23, a Lyapunov function for the system (6.14) and (6.15) can be chosen as

$$\overline{V}(\bar{x}_c, \bar{x}_s) = \frac{1}{2}\bar{x}_c^2 + \bar{x}_s^T P\, \bar{x}_s \tag{6.16}$$

Let the forward transformation be

$$\overline{x} = v(x) = x + v^2(x) + \cdots + v^{2k+1}(x). \tag{6.17}$$

Using (6.16), a Lyapunov function for the system (6.12) and (6.13) (in the original coordinates (x_c, x_s)) has the form:

$$\begin{aligned} V(x_c, x_s) &= \overline{V}(\overline{x})|_{\overline{x}=v(x)} \\ &= V^2(x_c, x_s) + \cdots + V^{2k+2}(x_c, x_s) + \cdots + V^{4k+2}(x_c, x_s). \end{aligned} \tag{6.18}$$

Let the function $V^{new}(x_c, x_s)$ be obtained from $V(x_c, x_s)$ by cutting terms $V^{2k+3}(x_c, x_s) + \cdots + V^{4k+2}(x_c, x_s)$, i.e.,

$$V^{new}(x_c, x_s) = V^2(x_c, x_s) + \cdots + V^{2k+1}(x_c, x_s) + V^{2k+2}(x_c, x_s). \tag{6.19}$$

We want to prove the following theorem.

Theorem 27 *The function*

$$V^{new}(x_c, x_s) = V^2(x_c, x_s) + \cdots + V^{2k+1}(x_c, x_s) + V^{2k+2}(x_c, x_s) \tag{6.20}$$

is a Lyapunov function for the system (6.12), (6.13). ◇◇◇

In order to prove this theorem, we need to introduce a key lemma first.

Consider a given function which is given as a summation of homogeneous polynomials:

$$\Pi(x) = \Pi^2(x) + \cdots + \Pi^k(x), \quad x \in R^n. \tag{6.21}$$

Assume that $\Pi(x)$ is locally positive definite, and furthermore, any analytic function $\tilde{\Pi}(x)$ of form

$$\tilde{\Pi}(x) = \Pi^2(x) + \cdots + \Pi^k(x) + O(|x|^{k+1}), \tag{6.22}$$

is also locally positive definite, where $O(|x|^{k+1})$ denotes any combinations of terms of degree $k+1$ or higher. Let

$$x = v(y) = y + v^2(y) + v^3(y) + \cdots$$

be an analytic transformation. It is obvious that

$$\begin{aligned} \tilde{\Lambda}(y) &\stackrel{def}{=} \Pi(x)|_{x=v(y)} = \overbrace{\Lambda^2(y) + \cdots + \Lambda^k(y)} + \Lambda^{k+1}(y) + \cdots \\ &\stackrel{def}{=} \Lambda(y) + \alpha(\|y\|^{k+1}) \end{aligned} \tag{6.23}$$

is still analytic and locally positive definite. Here, $\alpha(|y|^{k+1})$ denotes some combinations of terms of degree $k+1$ or higher. We claim that the following lemma holds.

Lemma 7 The function $\Lambda(y) = \Lambda^2(y) + \cdots + \Lambda^k(y)$ derived from (6.23) is locally positive definite. Furthermore, any analytic function $\hat{\Lambda}(y) = \Lambda(y) + O(|y|^{k+1}) = \Lambda^2(y) + \cdots + \Lambda^k(y) + O(|y|^{k+1})$ is also locally positive definite. ◇◇◇

Proof: Let the exact inverse of the analytic transformation $x = v(y) = y + v^2(y) + v^3(y) + \cdots$ be

$$y = u(x) = x + u^2(x) + u^3(x) + \cdots.$$

By the inverse function theorem [29], the analyticity of $v(y)$ at the origin $y = 0$ guarantees that of $u(x)$ at the origin $x = 0$. From $\Pi(x)\,|_{x=v(y)} = \Lambda(y) + \alpha(|y|^{k+1})$, we have

$$\Lambda(y) = \Pi(x) - \alpha(|y|^{k+1}) = \Pi(x) - \beta(|x|^{k+1}) \tag{6.24}$$

for some function $\beta(\cdot)$. Thus the local positive definiteness of function $\Lambda(y)$ follows from the local definiteness of function $\widetilde{\Pi}(x)$.

Notice that

$$\begin{aligned} \widehat{\Lambda}(y) &= \Lambda(y) + O(|y|^{k+1}) = \Pi(x) + O(|y|^{k+1}) - \beta(|x|^{k+1}) \\ &= \Pi(x) + \gamma(|x|^{k+1}) - \beta(|x|^{k+1}) \end{aligned} \tag{6.25}$$

for some function $\gamma(\cdot)$. The local positive definiteness of function $\widehat{\Lambda}(y)$ follows from the local definiteness of function $\widetilde{\Pi}(x)$ consequently. ◇◇◇

We are now ready to prove the Theorem 27.

Proof of the Theorem 27: Differentiating both sides of (6.18) with respect to time along the trajectory of the CASE 1 system (6.14) and (6.15) yields

$$\begin{aligned} \dot{V}(x_c, x_s) &= \dot{\overline{V}}(\bar{x}_c, \bar{x}_s)|_{\bar{x}=v(x)} \\ &= \left[a_{2k+1}\,\overline{x}_c^{2k+2} + \bar{x}_s^T \left((A^- + \Theta(\bar{x}_c, \bar{x}_s))^T P + P(A^- + \Theta(\bar{x}_c, \bar{x}_s))\right) \bar{x}_s \right. \\ &\quad \left. + O(|\overline{x}|^{2k+3})\right]_{\bar{x}=v(x)} \\ &= \left[\overline{W}^2(\bar{x}_c, \bar{x}_s) + \cdots + \overline{W}^{2k+2}(\bar{x}_c, \bar{x}_s) + O\left(|\overline{x}|^{2k+3}\right)\right]_{\bar{x}=v(x)} \\ &= W^2(x_c, x_s) + \cdots + W^{2k+2}(x_c, x_s) + O\left(|x|^{2k+2}\right). \end{aligned}$$

We have shown the local definiteness of both $\overline{W}^2(\bar{x}_c, \bar{x}_s) + \cdots + \overline{W}^{2k+2}(\bar{x}_c, \bar{x}_s)$ and $\overline{W}^2(\bar{x}_c, \bar{x}_s) + \cdots + \overline{W}^{2k+2}(\bar{x}_c, \bar{x}_s) + O\left(|\bar{x}|^{2k+3}\right)$. The local definiteness of both $W^2(x_c, x_s) + \cdots + W^{2k+2}(x_c, x_s)$ and $W^2(\bar{x}_c, \bar{x}_s) + \cdots + W^{2k+2}(x_c, x_s) + O\left(|x|^{2k+3}\right)$ follows directly from Lemma 7.

Since the terms $V^{2k+3}(x_c, x_s) + \cdots + V^{4k+2}(x_c, x_s)$ in $V(x_c, x_s)$ have no influence on the local negative definiteness of $\dot{V}(x_c, x_s)$, these terms can be truncated. Of course, $V^{new}(x_c, x_s)$ is still locally positive definite since its Hessian is a positive definite matrix. ◇◇◇

6.3 Examples

Example 21 (Continuation of example 19)

The following normal form transformation for the system (6.1)

$$\begin{aligned} x_1 &= \bar{x}_1 + c_1\bar{x}_1^2 - \bar{x}_1\bar{x}_2 + c_4\bar{x}_1^3 + (-c_1 - b_2)\,\bar{x}_1^2\bar{x}_2 + c_6\bar{x}_1\bar{x}_2^2, \\ x_2 &= \bar{x}_2 - 2\,\bar{x}_1^2 + b_2\bar{x}_1\bar{x}_2 + (1/2 - c_6)\,\bar{x}_2 \\ &\quad -4\,c_1\bar{x}_1^3 + b_5\bar{x}_1^2\bar{x}_2 + b_6\bar{x}_1\bar{x}_2^2 + b_7\bar{x}_2^3, \end{aligned}$$

yields a normal form up to degree 3

$$\begin{aligned} \dot{\bar{x}}_1 &= -\bar{x}_1^3, \\ \dot{\bar{x}}_2 &= -\bar{x}_2 + (1/2 - c_6)\bar{x}_2^2 + (b_6 - b_2(1/2 - c_6))\bar{x}_1\bar{x}_2^2 \\ &\quad +4\,\bar{x}_1^2\bar{x}_2 + \left(2\,b_7 - 2\,(1/2 - c_6)^2\right)\bar{x}_2^3, \end{aligned}$$

where the coefficients $\{c_1, c_4, c_6, b_2, b_5, b_6, b_7\}$ are free parameters. The forward transformations is

$$\begin{aligned}
\bar{x}_1 &= x_1 - c_1 x_1^2 + x_1 x_2 + \left(2 + 2\,{c_1}^2 - c_4\right) x_1^3 - 2\,c_1 x_1^2 x_2 + \frac{x_1 x_2^2}{2}, \\
\bar{x}_2 &= x_2 + 2\,{x_1}^2 - b_2 x_1 x_2 + (c_6 - 1/2)\,{x_2}^2 - 2\,b_2 {x_1}^3 \\
&\quad + \left(2 + {b_2}^2 + c_1 b_2 + 4\,c_6 - b_5\right) {x_1}^2 x_2 \\
&\quad + \left(-3\,c_6 b_2 + \frac{b_2}{2} - b_6\right) x_1 {x_2}^2 + \left(-2\,{c_6} + 1/2 + 2\,{c_6}^2 - b_7\right) {x_2}^3.
\end{aligned}$$

Choosing a Lyapunov function for this system in normal form as

$$\bar{V}(\bar{x}_1, \bar{x}_2) = \bar{x}_1^2 + \frac{\bar{x}_2^2}{16},$$

gives a Lyapunov function in the (x_1, x_2) coordinates as, after truncating terms of degree 5 or higher,

$$\begin{aligned}
V^{new}(x_1, x_2) &= x_1^2 + \frac{x_2^2}{16} - 2\,c_1 x_1^3 + \frac{9\,x_1^2 x_2}{4} - \frac{b_2 x_2^2 x_1}{8} + \left(\frac{c_6}{8} - \frac{1}{16}\right) x_2^3 \\
&\quad + \left(\frac{17}{4} + 5\,{c_1}^2 - 2\,c_4\right) x_1^4 + \left(-6\,c_1 - \frac{b_2}{2}\right) x_2 x_1^3 \\
&\quad + \left(\frac{17}{8} + \frac{3\,{b_2}^2}{16} + \frac{a_1 b_2}{8} + \frac{3\,c_6}{4} - \frac{b_5}{8}\right) x_2^2 x_1^2 \\
&\quad + \left(-\frac{3\,c_6 b_2}{8} + \frac{b_2}{16} - \frac{b_6}{8} - \frac{b_2\,(c_6 - 1/2)}{8}\right) x_2^3 x_1 \\
&\quad + \left(-\frac{c_6}{4} + \frac{1}{16} + \frac{{c_6}^2}{4} - \frac{b_7}{8} + \frac{(c_6 - 1/2)^2}{16}\right) x_2^4.
\end{aligned}$$

A very simple Lyapunov function

$$V^{new}(x_1, x_2) = x_1^2 + \frac{x_2^2}{16} + \frac{9\,x_1^2 x_2}{4}$$

is generated by setting $\{c_1 = 0, c_4 = 17/8, c_7 = 1/2, b_2 = 0, b_5 = 20, b_6 = 0, b_7 = -1/2\}$. The local negative definiteness of

$$\frac{dV^{new}(x_1, x_2)}{dt} = -\frac{x^2 y}{2} - \frac{5\,x^4}{2} + \frac{9\,x^2 y^2}{2} + \frac{9\,x^4 y}{2} - \frac{y^2}{8}.$$

can be easily verified Proposition 1. ◇◇◇

Example 22 (Continuation of Example 17 in Chapter 5)

Using Theorem 27, a simplified Lyapunov function for the CASE 2 nonlinear system (5.29) is given by:

$$\begin{aligned} V^{new}(x_1, x_2, x_3) &= {x_1}^2 + {x_2}^2 + {x_3}^2 + \frac{2\,{x_1}^3}{3} - 2\,{x_1}^2 x_2 + \frac{2\,{x_1}^2 x_3}{5} - \frac{2\,x_1 x_2 x_3}{5} \\ &- \frac{2\,{x_2}^2 x_3}{5} - \frac{4\,{x_2}^3}{3} + \frac{77\,{x_1}^4}{50} - \frac{29\,x_2 {x_1}^3}{50} + \frac{124\,{x_1}^2 {x_2}^2}{25} + \frac{2\,x_3 {x_1}^2 x_2}{5} \\ &+ {x_3}^2 {x_1}^2 + \frac{29\,x_1 {x_2}^3}{50} - \frac{x_3 x_1 {x_2}^2}{5} + \frac{127\,{x_2}^4}{50} + \frac{x_3 {x_2}^3}{5}. \end{aligned}$$

It is also easy to verify the local negative definiteness of

$$\begin{aligned} dV^{new}/dt &= -2\,{x_3}^2 + \frac{4\,{x_2}^2 x_3}{5} + \frac{4\,x_1 x_2 x_3}{5} - \frac{4\,{x_1}^2 x_3}{5} + \frac{4\,x_2 {x_1}^3}{25} \\ &- \frac{4\,x_1 {x_2}^3}{25} - \frac{29\,{x_2}^4}{50} - \frac{23\,{x_1}^2 {x_2}^2}{25} + \frac{4\,x_3 {x_1}^3}{5} - \frac{29\,{x_1}^4}{50} + O(|x|^5). \end{aligned}$$

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6.4 Summary

In this chapter, the procedure for the construction of Lyapunov functions for nonlinear systems was refined. If the local stability of a given nonlinear system $\dot{x} = Ax + f^2(x) + \cdots + f^{2k+1}(x) + \cdots$ is equivalent to that of the system $\dot{x} = Ax + f^2(x) + \cdots + f^{2k+1}(x)$, it was

shown that polynomial Lyapunov functions which include only terms of degree 2 to (2k+2) can be constructed. Since normal form transformations are not unique, the constructed Lyapunov functions can be further simplified by carefully choosing those free parameters in the normal form transformations. Several examples were provided to illustrate use of the refined algorithm.

CHAPTER 7
CONSTRUCTION OF LYAPUNOV FUNCTIONS - II

In this Chapter, we will provide an alternative algorithm for the construction of Lyapunov functions for critical CASE 1 and CASE 2 critical nonlinear systems. This algorithm requires that only the dynamics of critical states be transformed into a normal form up to an appropriate degree. In Section 7.1 a heuristic example is given. This example partially motivated the algorithm proposed in this chapter. Section 7.2 provides the main theorem of this chapter: a method for the construction of Lyapunov functions for any CASE 1 or CASE 2 nonlinear system once it is in an appropriate partial normal form. In Section 7.3, Lyapunov function in the original coordinates are derived, and simplified. The computation of partial normal forms is discussed in Section 7.4. More examples are presented in 7.5 for illustration of the algorithm. A brief summary in Section 7.5 concludes this chapter.

7.1 A Heuristic Example

Example 23 (A heuristic example)

Consider the system

$$\dot{x} = -x^3 + x^2y^2, \tag{7.1}$$

$$\dot{y} = -y + x^2 + y^3. \tag{7.2}$$

Since any center manifold must have the form $y = h(x) = O(|\cdot|^2)$, we can conclude, without actually calculating the center manifold, that the dynamics along the center manifold is

$$\dot{\xi} = -\xi^3 + O(|\cdot|^3)$$

which is locally asymptotically stable. Therefore, the entire system (7.1)-(7.2) is LAS.

Up to degree 3, an approximate normal form of (7.1)-(7.2) is given by:

$$\begin{aligned} \bar{x} &= -\bar{x}^3 + O(|\cdot|^4) \\ \bar{y} &= -\bar{y} + O(|\cdot|^4) \end{aligned}$$

with the associated transformations given by

$$\begin{aligned} x &= \bar{x} + O(|\cdot|^4), \\ y &= \bar{y} + \bar{x}^2 - \frac{\bar{y}^3}{2} + O(|\cdot|^4), \end{aligned}$$

and

$$\begin{aligned} \bar{x} &= x + O(|\cdot|^4) \\ \bar{y} &= y - x^2 + \frac{y^3}{2} + O(|\cdot|^4). \end{aligned}$$

The subsystem (7.1) can be viewed as in an approximate normal form, but the subsystem (7.2) is not. Nevertheless, without further applying either center manifold or normal form theory, the LASY of (7.1)-(7.2) is evident. This observation provides an inspiration for

the major results - the construction of Lyapunov functions using partial normal forms, presented in this chapter.

7.2 Main Results

For simplicity, consider an analytic nonlinear system described as follows:

$$\dot{x}_c = A^c x_c + f_c(x_c, x_s) = A^c x_c + f_c^2(x_c, x_s) + f_c^3(x_c, x_s) + \cdots, \tag{7.3}$$

$$\dot{x}_s = A^s x_s + f_s(x_c, x_s) = A^s x_s + f_s^2(x_c, x_s) + f_s^3(x_c, x_s) + \cdots. \tag{7.4}$$

Suppose that after applying an appropriate transformation of form

$$\bar{x}_c = x_c + v_1^2(x) + \cdots + v_1^k(x) \tag{7.5}$$

$$\bar{x}_s = x_s + v_2^2(x) + \cdots + v_2^k(x), \tag{7.6}$$

the dynamics for $(\bar{x}_c, \bar{x}_s)$ takes the form

$$\dot{\bar{x}}_c = A^c \bar{x}_c + g_c^2(\bar{x}_c) + g_c^3(\bar{x}_c) + \cdots + g_c^k(\bar{x}_c) + O(|\bar{x}|^{k+1}) \tag{7.7}$$

$$\dot{\bar{x}}_s = A^s \bar{x}_s + g_s^2(\bar{x}_c, \bar{x}_s) + g_s^3(\bar{x}_c, \bar{x}_s) + \cdots + g_s^k(\bar{x}_c, \bar{x}_s) + O(|\bar{x}|^{k+1}), \tag{7.8}$$

where equation (7.7) has the exact same dynamics as the critical dynamics of a normal form (up to degree k) for (7.3) and (7.4), but equation (7.8) does not. For a CASE 1 critical nonlinear system, the dynamics for $\bar{x}_c$ is

$$\dot{\bar{x}}_c = a_k \bar{x}^k + O(|\bar{x}|^{k+1}) \tag{7.9}$$

where a_k is the first nonzero real number in the sequence $\{a_2, a_3, a_4, \cdots\}$.

Similarly, for a CASE 2 critical nonlinear system, the dynamics for $\bar{x}_c$ takes the form

$$\begin{aligned}\dot{\bar{x}}_{c1} &= -\bar{x}_{c2} + (\bar{x}_{c1}^2 + \bar{x}_{c2}^2)(m_3\bar{x}_{c1} - n_3\bar{x}_{c2}) + \cdots \\ &\quad + (\bar{x}_{c1}^2 + \bar{x}_{c2}^2)^j(m_{(2j+1)}\bar{x}_{c1} - n_{(2j+1)}\bar{x}_{c2}) + O(|\bar{x}_c, \bar{x}_{c2}|^{2j+2}) \qquad (7.10)\\ \dot{\bar{x}}_{c2} &= \omega\bar{x}_{c1} + (\bar{x}_{c1}^2 + \bar{x}_{c2}^2)(m_3\bar{x}_{c2} + n_3\bar{x}_{c1}) + \cdots \\ &\quad + (\bar{x}_{c1}^2 + \bar{x}_{c2}^2)^j(m_{(2j+1)}\bar{x}_{c2} + n_{(2j+1)}\bar{x}_{c1}) + O(|\bar{x}_c, \bar{x}_s|^{(2j+2)}), \qquad (7.11)\end{aligned}$$

where $k = 2j + 1$ is odd, and m_k is the first nonzero real number in the sequence $\{m_3, m_5, m_7, \cdots\}$.

We will prove the following theorem by the explicit construction of Lyapunov function candidates.

Theorem 28 *For any CASE 1 system described by (7.9), (7.8), it is LAS when $a_k < 0$ and k is odd, and unstable otherwise. For any CASE 2 system described by (7.10), (7.11) and (7.8), it is LAS when $m_{2j+1} < 0$, and unstable otherwise.* ◇◇◇

Before proving this theorem, we need the following lemmas.

Lemma 8 *Let a homogeneous polynomial function $\Pi^k(x) : R^n \to R$ be positive definite, where $k > 0$ is an even integer. Then any analytic function $\Pi(x) : R^n \to R$ defined by*

$$\Pi(x) = \Pi^k(x) + O(|x|^{k+1}), \qquad (7.12)$$

is also locally positive definite. ◇◇◇

Proof: Let Ω be a neighborhood of the origin. Let $0 \neq x \in \Omega$, and $t \in R$ with $0 < |t| < 1$. By Taylor's theorem there exists a real parameter $0 < \xi < 1$ such that

$$\Pi(tx) = \frac{1}{k!}\frac{\partial^k \Pi(y)}{\partial y^k}|_{y=\xi tx}(tx)^k = \frac{t^k}{k!}\frac{\partial^k \Pi(y)}{\partial y^k}|_{y=\xi tx}(x)^k. \tag{7.13}$$

Consider the function $\Gamma(t) \triangleq \frac{1}{k!}\frac{\partial^k \Pi(y)}{\partial y^k}|_{y=tx}(x)^k$. The function $\Gamma(t)$ is locally continuous since $\Pi(x)$ is C^ω. Since $\Gamma(0) = \frac{1}{k!}D^k\Pi(0)(x)^k = \Pi^k(x) > 0$, by continuity there exist a real number $\varepsilon > 0$ such that for all $|t| < \varepsilon$, $\Gamma(t) > 0$. This means that for all $|t| < \min\{1, \varepsilon\}$, $\frac{1}{k!}D^k\Pi(tx)(x)^k > 0$. Since $0 < \xi < 1$, we have $\frac{1}{k!}D^k\Pi(\xi tx)(x)^k > 0$. Because k is an even number, $\frac{t^k}{k!}D^k\Pi(\xi tx)(x)^k = \Pi(tx) > 0$ whenever $t \neq 0$ and $x \neq 0$. Thus $\Pi(x)$ is locally positive definite. ◇◇◇

Lemma 9 Consider a linear system $\dot{x} = Ax$. If it is asymptotically stable, and $V(x) = x^T Px$ is a Lyapunov function, then for any positive integer k, $W(x) = (x^T Px)^k$ is a Lyapunov function for any analytic nonlinear system $\dot{x} = Ax + O(|x|^2)$.

Proof: The positive definiteness of $W(x)$ is evident. Differentiating $W(x)$ respect to time yields

$$\frac{dW(x)}{dt} = k(x^T Px)^{k-1}\frac{dV(x)}{dt} = k(x^T Px)^{k-1}\left(x^T(A^T P + PA)x + O(|x|^3)\right)$$

Since $(x^T Px)^{k-1}$ is positive definite and $\frac{dV(x)}{dt}$ is locally negative definite, $\frac{dW(x)}{dt}$ is locally negative definite. ◇◇◇

We are now ready to prove the main theorem in this chapter.

Proof of Theorem 28: Since a CASE 2 system is equivalent to a CASE 1 system after using polar coordinates, theorem (28) will be only proved for CASE 1 systems.

- $a_k < 0$ and k is odd: Choose $\overline{V}(\bar{x}_c, \bar{x}_s) = \frac{1}{2}\bar{x}_c^2 + \frac{2}{k+1}\left(\bar{x}_s^T P \bar{x}_s\right)^{(k+1)/2}$ which is positive definite. Then

$$\begin{aligned}\frac{d\overline{V}(\bar{x}_c, \bar{x}_s)}{dt} &= \bar{x}_c\left(a_k\bar{x}_c^k + O(|\bar{x}|^{k+1})\right) + \left(\bar{x}_s^T P \bar{x}_s\right)^{(k-1)/2}\left(\bar{x}_s^T Q\bar{x}_s + O(|\bar{x}|^3)\right) \\ &= a_k\bar{x}_c^{k+1} + \left(\bar{x}_s^T P \bar{x}_s\right)^{(k-1)/2}\left(\bar{x}_s^T Q\bar{x}_s\right) + O(|\bar{x}|^{k+2}).\end{aligned}$$

where $Q = (A^{sT}P + PA^s)$. The local negative definiteness of $\frac{d\overline{V}(\bar{x}_c,\bar{x}_s)}{dt}$ follows since $\Pi^{k+1}(\bar{x}_c, \bar{x}_s) \triangleq a_k\bar{x}_c^{k+1} + \left(\bar{x}_s^T P \bar{x}_s\right)^{(k-1)/2}\left(\bar{x}_s^T Q\bar{x}_s\right)$ is negative definite.

- $a_k > 0$ and k is odd: Choose $\overline{V}(\bar{x}_c, \bar{x}_s) = -\frac{1}{2}\bar{x}_c^2 + \frac{2}{k+1}\left(\bar{x}_s^T P \bar{x}_s\right)^{(k+1)/2}$ which is locally indefinite.

$$\frac{d\overline{V}(\bar{x}_c, \bar{x}_s)}{dt} = -a_k\bar{x}_c^{k+1} + \left(\bar{x}_s^T P \bar{x}_s\right)^{(k-1)/2}\left(\bar{x}_s^T Q\bar{x}_s\right) + O(|\bar{x}|^{k+2})$$

is locally negative definite.

- $a_k \neq 0$ and k is even: Choose $\overline{V}(\bar{x}_c, \bar{x}_s) = a_k\bar{x}_c + \frac{2}{k}\left(\bar{x}_s^T P \bar{x}_s\right)^{k/2}$ which is indefinite.

$$\frac{d\overline{V}(\bar{x}_c, \bar{x}_s)}{dt} = -a_k^2\bar{x}_c^k + \left(\bar{x}_s^T P \bar{x}_s\right)^{k/2-1}\left(\bar{x}_s^T Q\bar{x}_s\right) + O(|\bar{x}|^{k+1})$$

is locally negative definite.

These observations prove Theorem 28 for CASE 1 systems.

For CASE 2 systems, the local asymptotic stability or instability can be proved by using $\overline{V}(\bar{x}_c, \bar{x}_s) = \pm\frac{1}{2}(\bar{x}_{c1}^2 + \bar{x}_{c2}^2) + \frac{2}{k+1}\left(\bar{x}_s^T P \bar{x}_s\right)^{(k+1)/2}$. ◇◇◇

Lyapunov functions have been constructed for CASE 1 or CASE 2 nonlinear systems in partial normal forms. One of the properties of these Lyapunov functions is that their

time derivatives along the system dynamics are negative definite homogeneous polynomial functions plus terms of higher degree. Lyapunov functions in the original coordinates (x_c, x_s) can be obtained by just substituting the transformation $\bar{x} = v(x)$ which was used to obtain the partial normal forms, i.e.,

$$V(x_c, c_s) = \bar{V}(\bar{x}_c, \bar{x}_s) \mid_{\bar{x}=v(x)} .$$

In the next section, we will propose a method for simplification of Lyapunov functions in the original coordinates.

7.3 Simplification of Lyapunov Functions

In this section we address how to simplify the Lyapunov functions constructed using the algorithm proposed in Section 7.2.

For an LAS CASE 1 nonlinear system, a Lyapunov function in partial normal form coordinates $(\bar{x}_c, \bar{x}_s)$ constructed in previous section is given by

$$\overline{V}(\bar{x}_c, \bar{x}_s) = \frac{1}{2}\bar{x}_c^2 + \frac{1}{k+1}\left(\bar{x}_s^T P\, \bar{x}_s\right)^{(k+1)},$$

where a_{2k+1} is the first nonzero real number in the sequence $\{a_2, a_3, a_4, \cdots\}$.

Similarly, for an LAS CASE 2 nonlinear system, a Lyapunov function in partial normal form coordinates $(\bar{x}_c, \bar{x}_s)$ constructed in previous section is given by

$$\overline{V}(\bar{x}_c, \bar{x}_s) = \frac{1}{2}(\bar{x}_{c,1}^2 + \bar{x}_{c,2}^2) + \frac{1}{k+1}\left(\bar{x}_s^T P\, \bar{x}_s\right)^{(k+1)},$$

where m_{2k+1} is the first nonzero real number in the sequence $\{m_3, a_5, a_7, \cdots\}$.

A Lyapunov function in the original coordinates (x_c, x_s) is obtained by just substituting the forward transformation

$$\bar{x} = v(x) = x + v^2(x) + \cdots + v^{2k+1}(x),$$

into $\bar{V}(\bar{x})$ to give

$$\begin{aligned} V(x_c, x_s) &= \overline{V}(\bar{x}_c, \bar{x}_s)|_{\bar{x}=v(x)} \\ &= V^2(x_c, x_s) + \cdots + V^{2k+2}(x_c, x_s) + \cdots + V^{(2k+1)(k+1)}(x_c, x_s), \end{aligned} \tag{7.14}$$

which is quite complicated, and includes terms of degree 2 to $(2k+1)(k+1)$. Consider now the time derivative of (7.14) along the dynamics

$$\begin{aligned} \frac{dV(x_c, x_s)}{dt} &= \frac{d\overline{V}(\bar{x}_c, \bar{x}_s)}{dt}|_{\bar{x}=v(x)} = \left(\Pi^{2k+2}(\bar{x}_c, \bar{x}_s) + O(|\bar{x}|^{2k+3})\right)_{\bar{x}=v(x)} \\ &= \Pi^{2k+2}(x_c, x_s) + O(|x|^{2k+3}), \end{aligned} \tag{7.15}$$

for some homogeneous polynomial function of degree $2k+2$: $\Pi^{2k+2}(x_c, x_s)$. Notice that term $V^{2k+3}(x_c, x_s) + \cdots + V^{(2k+1)(k+1)}(x_c, x_s)$ in (7.14) only contributes terms of degree $2k+3$ or higher in (7.15). Let function $V^{new}(x_c, x_s)$ be obtained by truncating the terms of degree $2k+3$ or higher in (7.14), i.e.,

$$V^{new}(x_c, x_s) = V^2(x_c, x_s) + V^3(x_c, x_s) + \cdots + V^{2k+2}(x_c, x_s). \tag{7.16}$$

The time derivative of $V^{new}(x_c, x_s)$ will also take the form:

$$\frac{dV^{new}(x_c, x_s)}{dt} = \Pi^{k+1}(x_c, x_s) + O(|x|^{k+2}). \tag{7.17}$$

The following theorem is obtained due to this observation.

Theorem 29 *Assume that a CASE 1 or CASE 2 critical nonlinear system $\dot{x} = f(x)$ is LAS, and $a_{2k+1} < 0$ (or $m_{2k+1} < 0$) is the first nonzero real number in the sequence $\{a_2, a_3, a_4, \cdots\}$ (or $\{m_3, m_5, m_7, \cdots\}$). If $V(x_c, x_s)$ constructed as in (7.14) is a Lyapunov function for $\dot{x} = f(x)$, then $V^{new}(x_c, x_s) = V^2(x_c, x_s) + V^3(x_c, x_s) + \cdots + V^{2k+2}(x_c, x_s)$ is a Lyapunov function.* ◇◇◇

Proof: The local negative definiteness of $dV^{new}(x_c, x_s)/dt$ is already proved. We need to prove that $V^{new}(x_c, x_s)$ is locally positive definite. We will prove it by contradiction. Remember that locally, a function can only be indefinite, positive semi-definite, positive definite, negative semi-definite, or negative definite.

- Assume that $V^{new}(x_c, x_s)$ is locally indefinite (or negative definite, or negative semi-definite). Therefore, there exists a region arbitrarily close to the origin in which both $dV^{new}(x_c, x_s)/dt$ and $V^{new}(x_c, x_s)$ are negative. This implies that the system is unstable using an instability theorem given in [37]. That is a contradiction.

- Assume that $V^{new}(x_c, x_s)$ is positive semi-definite and $dV^{new}(x_c, x_s)/dt$ negative in an open neighborhood Ω of the origin ($x_c = 0, x_s = 0$). Then there exists a point $x^0 \in \Omega$ such that $V^{new}(x_c^0, x_s^0) = 0$. By assumption, there is an open neighborhood $\Delta \subset \Omega$ of x^0 such that for any $y \in \Delta \subset \Omega$, $V^{new}(y_c, y_s) \geq 0$. However $dV^{new}(y_c, y_s)/dt < 0$

implies that there exists an point $z \in \Delta \subset \Omega$ such that $V^{new}(z_c, z_s) < 0$. This is alsoy a contradiction.

In summary, $V^{new}(x_c, x_s) = V^2(x_c, x_s) + V^3(x_c, x_s) + \cdots + V^{2k+2}(x_c, x_s)$ must be locally positive definite, and hence a Lyapunov function. $\diamond\diamond\diamond$

7.4 Computation of Partial Normal Forms

In this section, we address the issues related to the computation of partial normal forms and the associated transformations. Let the original system, the transformed system, transformations and the PDE be given respectively as follows:

$$\begin{pmatrix} \dot{x}_c \\ \dot{x}_s \end{pmatrix} = \begin{pmatrix} f_c(x,a) \\ f_s(x,a) \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} f_c^{m+1}(x) \\ f_s^{m+1}(x) \end{pmatrix} \frac{a^m}{m!}, \tag{7.18}$$

$$\begin{pmatrix} \dot{y}_c \\ \dot{y}_s \end{pmatrix} = \begin{pmatrix} g_c(y,a) \\ g_s(y,a) \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} g_c^{m+1}(y) \\ g_s^{m+1}(y) \end{pmatrix} \frac{a^m}{m!}, \tag{7.19}$$

$$\begin{pmatrix} x_c \\ x_s \end{pmatrix} = \begin{pmatrix} u_c(y,a) \\ u_s(y,a) \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} u_c^{m+1}(y) \\ u_s^{m+1}(y) \end{pmatrix} \frac{a^m}{m!}, \; u(y,0) = y, \tag{7.20}$$

$$\begin{pmatrix} y_c \\ y_s \end{pmatrix} = \begin{pmatrix} v_c(x,a) \\ v_s(x,a) \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} v_c^{m+1}(x) \\ v_s^{m+1}(x) \end{pmatrix} \frac{a^m}{m!}, \; v(x,0) = x, \tag{7.21}$$

$$\begin{pmatrix} \frac{\partial u_c}{\partial a} \\ \frac{\partial u_s}{\partial a} \end{pmatrix} = \begin{pmatrix} U_c(u,a) \\ U_s(u,a) \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} U_c^{m+2}(u) \\ U_s^{m+2}(u) \end{pmatrix} \frac{a^m}{m!}, \; u(y,0) = y. \tag{7.22}$$

The terms of degree m, g^m, f^m, and U^m are related by the following equation, where Γ^m only depends on $f^1, \cdots, f^m$ and $U^2, \cdots, U^{m-1}$,

$$g^m(x) = \Gamma^m(x) - L_A^m U^m(x)$$

$$
\begin{aligned}
&= \Gamma^m(x) - \left(\frac{\partial U^m(x)}{\partial x}Ax - AU^m(x)\right) \\
&= \begin{pmatrix} \Gamma_c^m(x) \\ \Gamma_s^m(x) \end{pmatrix} - \left\{ \begin{pmatrix} \frac{\partial U_c^m}{\partial x_c} & \frac{\partial U_c^m}{\partial x_s} \\ \frac{\partial U_s^m}{\partial x_c} & \frac{\partial U_s^m}{\partial x_s} \end{pmatrix} \begin{pmatrix} A^c x_c \\ A^s x_s \end{pmatrix} - \begin{pmatrix} A^c U_c^m \\ A^s U_s^m \end{pmatrix} \right\} \\
&= \begin{pmatrix} \Gamma_c^m(x) \\ \Gamma_s^m(x) \end{pmatrix} - \begin{pmatrix} \frac{\partial U_c^m}{\partial x_c}A^c x_c + \frac{\partial U_c^m}{\partial x_s}A^s x_s - A^c U_c^m(x) \\ \frac{\partial U_s^m}{\partial x_c}A^c x_c + \frac{\partial U_s^m}{\partial x_s}A^s x_s - A^s U_s^m(x) \end{pmatrix}.
\end{aligned}
$$

It is clear that the normalization of $g_c^m(x)$ depends on the choice of $U_s^m(x)$. Therefore we can simply set $U_s^m(x) = 0$ to simplify the computation of partial normal forms. Using this construction, Theorem 12 and Theorem 13 in Chapter 3 can be reduced to the following forms.

Theorem 30 *Let $f(x,a)$, $g(y,a)$ and $U(u,a)$ be expanded as above, and take f_i (or g_i, or U_i) to be f^{i+1} (or g^{i+1}, or U^{i+2}), respectively. The sequence $\{g_{c,i}^T, g_{s,i}^T\}$ may be computed as follows:*

$$
\begin{pmatrix} g_{c,i}(y) \\ g_{s,i}(y) \end{pmatrix} = \begin{pmatrix} f_{c,0}^{(i)}(y) \\ f_{s,0}^{(i)}(y) \end{pmatrix},
$$

where the sequence $f_{i-m}^{(m)}$ is defined recursively by

$$
\begin{pmatrix} f_{c,i}^{(0)} \\ f_{s,i}^{(0)} \end{pmatrix} = \begin{pmatrix} f_{c,i} \\ f_{s,i} \end{pmatrix},
$$

and for $m = 1, 2, 3, \cdots, i$,

$$
\begin{pmatrix} f_{c,i-m}^{(m)}(x) \\ f_{s,i-m}^{(m)}(x) \end{pmatrix} = \begin{pmatrix} f_{c,i-m+1}^{(m-1)}(x) \\ f_{s,i-m+1}^{(m-1)}(x) \end{pmatrix}
$$

$$+\sum_{j=0}^{i-m}\binom{i-m}{j}\begin{pmatrix}\frac{\partial f_{c,\,i-m+1}^{(m-1)}(x)}{\partial x_c}U_{c,\,j}(x)-\frac{\partial U_{c,\,j}(x)}{\partial x_c}f_{c,\,i-m-j}^{(m-1)}(x)-\frac{\partial U_{c,\,j}(x)}{\partial x_s}f_{s,\,i-m-j}^{(m-1)}(x)\\ \frac{\partial f_{s,\,i-m+1}^{(m-1)}(x)}{\partial x_c}U_{c,\,j}(x)\end{pmatrix}.$$

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Theorem 31 *Assume $u(y,a)$ is the solution of the PDE, and*

$$p(x,a)=\begin{pmatrix}p_c(x,a)\\ p_s(x,a)\end{pmatrix}=\sum_{m=0}^{\infty}\begin{pmatrix}p_{c,\,m}(x)\\ p_{s,\,m}(x)\end{pmatrix}\frac{a^m}{m!}. \tag{7.23}$$

Let

$$q(y,a)=\begin{pmatrix}q_c(y,a)\\ q_s(y,a)\end{pmatrix}=p(u(y,a),a)=\sum_{m=0}^{\infty}\begin{pmatrix}q_{c,\,m}(y)\\ q_{s,\,m}(y)\end{pmatrix}\frac{a^m}{m!}. \tag{7.24}$$

The sequence $\{q_{c,i}^T, q_{s,i}^T\}$ may be computed as follows:

$$q_i(y)=\begin{pmatrix}q_{c,\,i}(y)\\ q_{s,\,i}(y)\end{pmatrix}=p_0^{(i)}(y)=\begin{pmatrix}p_{c,0}^{(i)}(y)\\ p_{s,0}^{(i)}(y)\end{pmatrix},$$

where the sequence $p_{i-m}^{(m)}$ is defined recursively by,

$$\begin{aligned}p_i^{(0)} &= \begin{pmatrix}p_{c,\,i}^{(0)}\\ p_{s,\,i}^{(0)}\end{pmatrix}=p_i=\begin{pmatrix}p_{c,\,i}\\ p_{s,\,i}\end{pmatrix}\\ p_{i-m}^{(m)}(x) &= \begin{pmatrix}p_{c,\,i-m}^{(m)}(x)\\ p_{s,\,i-m}^{(m)}(x)\end{pmatrix}=\begin{pmatrix}p_{c,\,i-m+1}^{(m-1)}(x)\\ p_{s,\,i-m+1}^{(m-1)}(x)\end{pmatrix}\\ &+\sum_{j=0}^{i-m}\binom{i-m}{j}\begin{pmatrix}\frac{\partial p_{c,\,i-m+1}^{(m-1)}(x)}{\partial x_c}\\ \frac{\partial p_{s,\,i-m+1}^{(m-1)}(x)}{\partial x_c}\end{pmatrix}U_{c,\,j}(x),\ m=1,2,\cdots,i.\end{aligned}$$

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The algorithms for deriving the transformations u^m, v^m, can be easily written out accordingly. With a little bit more effort, we can easily verify that $u_s^m = v_s^m = 0$ if $U_c^m = 0$ for $m = 2, 3, \cdots$. The first few elements in the sequence $\{u_c^2,\ u_c^3,\ u_c^4,\ \cdots\}$ and $\{v_c^2,\ v_c^3,\ v_c^4,\ \cdots\}$ can be written out as follows:

$$
\begin{aligned}
u_c^2(y) &= U_c^2(y) \\
u_c^3(y) &= U_c^3(y) + \frac{\partial U_c^2(y)}{\partial y_c} U_c^2(y) \\
u_c^4(y) &= U_c^4(y) + \frac{\partial U_c^3(y)}{\partial y_c} U_c^2(y) + \frac{\partial U_c^2(y)}{\partial y_c} U_c^3(y) + \frac{\partial u_c^3(y)}{\partial y_c} U_c^2(y) \\
u_c^5(y) &= U_c^5(y) + \frac{\partial U_c^4(y)}{\partial y_c} U_c^2(y) + 2\frac{\partial U_c^3(y)}{\partial y_c} U_c^3(y) \\
&\quad + \frac{\partial}{\partial y_c}\left(U_c^4(y) + \frac{\partial U_c^3(y)}{\partial y_c} U_c^2(y) + \frac{\partial U_c^2(y)}{\partial y_c} U_c^3(y) \right) U_c^2(y) \\
&\quad + \frac{\partial u_c^3(y)}{\partial y_c} U_c^3(y) + \frac{\partial u_c^4(y)}{\partial y_c} U_c^2(y) + \frac{\partial U_c^2(y)}{\partial y_c} U_c^4(y).
\end{aligned}
$$

and

$$
\begin{aligned}
v_c^2(y) &= -U_c^2(y) \\
v_c^3(y) &= -\frac{\partial v_c^2(y)}{\partial y_c} U_c^2(y) - U_c^3(y) \\
v_c^4(y) &= -\frac{\partial v_c^3(y)}{\partial y_c} U_c^2(y) - 2\frac{\partial v_c^2(y)}{\partial y_c} U_c^3(y) - U_c^4(y) \\
v_c^5(y) &= -\frac{\partial v_c^4}{\partial y} U_c^2 - 3\frac{\partial v_c^3}{\partial y} U_c^3 - 3\frac{\partial v_c^2}{\partial y} U_c^4 - U_c^5
\end{aligned}
$$

Remark 10 *The advantage of the computation of partial normal forms and the associated transformations can be observed from the discussions above. Assume x_s and x_c have the same dimension. The amount of time necessary to compute partial normal forms and the*

associated transformations is about half of the time necessary for computing normal forms and the associated transformations for all states. Actually, the dimension of x_c is much lower than the dimension of x_s in many applications. ◇◇◇

7.5 Examples

In this section, examples are provided to illustrate the use of the algorithm proposed in this chapter.

Example 24 (Reconsideration of Example 23)

Consider again the system

$$\dot{x} = -x^3 + x^2y^2, \tag{7.25}$$

$$\dot{y} = -y + x^2 + y^3. \tag{7.26}$$

A Lyapunov function is given by

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{4}y^4$$

with

$$\frac{dV(x,y)}{dt} = -(x^4 + y^4) + O(|x,y|^5).$$

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Example 25 (The CASE 1 system in Example 16 in Chapter 5)

Consider the system

$$\dot{x} = -xy + xz^2 + xyz,$$

$$\begin{aligned} \dot{y} &= -y + x^2 + xz^2, \\ \dot{z} &= -z + xy. \end{aligned}$$

A partial normal form (up to degree 3) is

$$\begin{aligned} \dot{\bar{x}} &= -\bar{x}^3, \\ \dot{\bar{y}} &= -\bar{y} + \bar{x}^2 + \bar{x}\bar{z}^2 + 2\,\bar{x}^2\bar{y}, \\ \dot{\bar{z}} &= -\bar{z} + \bar{x}\bar{y} + \bar{x}\bar{y}^2. \end{aligned}$$

The associated backward transformation is

$$\begin{aligned} x &= \bar{x} + \bar{x}\bar{y} - \frac{\bar{x}\bar{y}\bar{z}}{2} - \frac{\bar{x}\bar{z}^2}{2} + \frac{\bar{x}\bar{y}^2}{2}, \\ y &= \bar{y}, \\ z &= \bar{z}. \end{aligned}$$

The associated forward transformation is

$$\begin{aligned} \bar{x} &= x - xy + \frac{xy^2}{2} + \frac{xyz}{2} + \frac{xz^2}{2}, \\ \bar{y} &= y, \\ \bar{z} &= z. \end{aligned}$$

The following Lyapunov function

$$V(x, y, z) = x^2 - 2\,x^2y + z^2x^2 + 2\,x^2y^2 + yzx^2 + y^4 + z^4$$

is derived by truncating terms of degree of 5 or higher using $\bar{V}(\bar{x},\bar{y},\bar{z}) = \bar{x}^2+\bar{y}^4+\bar{z}^4$. $\diamond\diamond\diamond$

Example 26 (The CASE 2 system in Example 17 in Chapter 5)

Consider the system

$$\begin{aligned}
\dot{x} &= -y + xz^2 + xyz + x^2 + z^2, \\
\dot{y} &= x + z^2 + x^2 + xz^2, \\
\dot{z} &= -z + xy + z^2.
\end{aligned}$$

A partial normal form (up to degree 3) is

$$\begin{aligned}
\dot{\bar{x}} &= -\bar{y} - \frac{\bar{x}^3}{4} + \frac{7\,\bar{x}^2\bar{y}}{12} - \frac{\bar{x}\bar{y}^2}{4} + \frac{7\,\bar{y}^3}{12} \\
&= -\bar{y} - \frac{(3\,\bar{x} - 7\,\bar{y})\,(\bar{x}^2+\bar{y}^2)}{12}, \\
\dot{\bar{y}} &= \bar{x} - \frac{7\,\bar{x}^3}{12} - \frac{\bar{x}^2\bar{y}}{4} - \frac{7\,\bar{x}\bar{y}^2}{12} - \frac{\bar{y}^3}{4} \\
&= \bar{x} - \frac{(7\,\bar{x} + 3\,\bar{y})\,(\bar{x}^2+\bar{y}^2)}{12}, \\
\dot{\bar{z}} &= -\bar{z} + \bar{x}\bar{y} + \bar{z}^2 + \frac{\bar{x}^2\bar{y}}{3} + \frac{4\,\bar{x}\bar{y}^2}{3} - \frac{2\,\bar{y}^3}{3} - \frac{3\,\bar{y}\bar{z}^2}{5} + \frac{\bar{x}^3}{3} - \frac{\bar{x}\bar{z}^2}{5}.
\end{aligned}$$

The associated backward transformation (up to degree 3) is

$$\begin{aligned}
x &= \bar{x} - \frac{\bar{x}^2}{3} + \frac{2\,\bar{x}\bar{y}}{3} - \frac{2\,\bar{y}^2}{3} - \frac{3\,\bar{z}^2}{5} - \frac{7\,\bar{x}^3}{36} - \frac{\bar{x}^2\bar{y}}{2} - \frac{12\,\bar{x}^2\bar{z}}{25} \\
&\quad - \frac{31\,\bar{x}\bar{y}\bar{z}}{50} + \frac{\bar{x}\bar{z}^2}{10} - \frac{11\,\bar{y}^3}{12} + \frac{12\,\bar{y}^2\bar{z}}{25} - \frac{\bar{y}\bar{z}^2}{20} - \frac{2\,\bar{z}^3}{5} + \frac{2\,\bar{x}\bar{y}^2}{9}, \\
y &= \bar{y} + \frac{\bar{x}^2}{3} + \frac{2\,\bar{x}\bar{y}}{3} + \frac{2\,\bar{y}^2}{3} - \frac{\bar{z}^2}{5} + \frac{\bar{x}^3}{12} - \frac{5\,\bar{x}^2\bar{y}}{18} + \frac{7\,\bar{x}^2\bar{z}}{50} \\
&\quad - \frac{17\,\bar{x}\bar{y}\bar{z}}{50} + \frac{\bar{x}\bar{z}^2}{20} - \frac{13\,\bar{y}^3}{36} - \frac{7\,\bar{y}^2\bar{z}}{50} + \frac{2\,\bar{x}\bar{y}^2}{3}, \\
z &= \bar{z}.
\end{aligned}$$

The associated forward transformation is

$$\begin{aligned}
\bar{x} &= x + \frac{x^2}{3} - \frac{2xy}{3} + \frac{2y^2}{3} + \frac{3z^2}{5} + \frac{23x^3}{36} - \frac{x^2y}{6} + \frac{12x^2z}{25} \\
&\quad + \frac{2xy^2}{9} + \frac{31xyz}{50} + \frac{xz^2}{6} - \frac{5y^3}{12} - \frac{12y^2z}{25} - \frac{yz^2}{12} + \frac{2z^3}{5}, \\
\bar{y} &= y - \frac{x^2}{3} - \frac{2xy}{3} - \frac{2y^2}{3} + \frac{z^2}{5} - \frac{x^3}{12} + \frac{25x^2y}{18} - \frac{7x^2z}{50} \\
&\quad + \frac{2xy^2}{3} + \frac{17xyz}{50} - \frac{7xz^2}{12} + \frac{29y^3}{36} + \frac{7y^2z}{50} - \frac{2yz^2}{3}, \\
\bar{z} &= z.
\end{aligned}$$

The following Lyapunov function

$$\begin{aligned}
V(x,y,z) &= x^2 + y^2 + \frac{2x^3}{3} - 2x^2y + \frac{6xz^2}{5} - \frac{4y^3}{3} + \frac{2yz^2}{5} + \frac{3x^4}{2} \\
&\quad + \frac{24zx^3}{25} - \frac{yx^3}{2} + 5x^2y^2 + \frac{24zx^2y}{25} + \frac{3z^2x^2}{5} + \frac{xy^3}{2} \\
&\quad - \frac{7zxy^2}{25} - \frac{12z^2xy}{5} + \frac{4z^3x}{5} + \frac{5y^4}{2} + \frac{7zy^3}{25} - \frac{4z^2y^2}{5} + \frac{7z^4}{5}
\end{aligned}$$

is derived using $\bar{V}(\bar{x},\bar{y},\bar{z}) = \bar{x}^2 + \bar{y}^2 + \bar{z}^4$. ◊◊◊

7.6 Summary

In this chapter, an alternative method is proposed for the construction of Lyapunov functions for CASE 1 and CASE 2 critical nonlinear systems. This method requires deriving only a partial normal form up to an appropriate degree so that the computation is reduced significantly. After such a partial normal form is obtained, a Lyapunov function for the system in the partial normal form coordinates is constructed easily. A Lyapunov function for the system in the original coordinates is obtained by just substituting of the associated forward transformation. This Lyapunov function can be further simplified by truncating some of the terms of higher degrees.

CHAPTER 8
APPLICATIONS

Two different algorithms for the construction of Lyapunov functions for CASE 1 and CASE 2 critical nonlinear systems in were developed in Chapters 5 through 7. In Section 8.1 of this chapter, nonlinear systems with multiple critical eigenvalues are studied using normal form techniques. This dissertation focuses on continuous-time nonlinear systems. Normal form techniques are also applicable to discrete-time nonlinear systems with only slight modifications. An introduction to normal forms for discrete-time nonlinear systems is given in Section 8.2. Finally, a brief summary of this chapter is given in 8.3.

8.1 Nonlinear Systems with Multiple Critical Eigenvalues

In this section we study critical nonlinear systems with more than one zero eigenvalue or more than one pair of purely imaginary eigenvalues using normal form techniques. Sufficient conditions for the LASY of such nonlinear systems will be derived. Let us look at an example first.

Example 27 *Consider a critical nonlinear system as follows:*

$$\begin{aligned}
\dot{x} &= -y - y^2 + 2xy - 4x^3 - \frac{5}{2}x^2y - 2xy^2 - \frac{9}{2}y^3 \\
&\quad -4x^4 + \frac{3}{2}x^2y^2 + xy^3 - \frac{23}{2}y^4 - 4x^5 - \frac{323}{12}y^5 \\
&\quad +\frac{4}{3}x^4y + 4x^3y^2 + \frac{23}{4}x^2y^3 + 7xy^4 \qquad (8.1) \\
\dot{y} &= x + x^2 - 2xy + \frac{9}{2}x^3 - 6x^2y + \frac{5}{2}xy^2 - 4y^3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{23}{2}x^4 - 17\,x^3y + \frac{13}{2}\,x^2y^2 - 4\,y^4 + \frac{323}{12}\,x^5 \\
&-4\,y^5 - 39\,x^4y + \frac{41}{4}x^3y^2 + 4\,x^2y^3 - \frac{4}{3}\,xy^4 \qquad (8.2)\\
\dot{z} \;=\;& y^2 - x^2 - xy^2 - \frac{3\,x^3}{2} + x^2z - y^2z + x^2y + \frac{3\,y^3}{2} \\
&-\frac{11\,x^4}{2} - \frac{y^2z^2}{6} - \frac{5\,y^3z}{3} - \frac{2\,x^2yz}{3} + \frac{19\,xy^3}{3} \\
&+\frac{11\,y^4}{2} + \frac{29\,x^3y}{3} + \frac{2\,xy^2z}{3} + \frac{x^2z^2}{6} + \frac{5\,x^3z}{3} \\
&-\frac{119\,x^5}{8} - \frac{x^2yz^2}{12} + \frac{25\,x^2y^3}{8} - \frac{71\,y^4z}{12} + \frac{49\,xy^4}{4} \\
&+\frac{119\,y^5}{8} - \frac{43\,xy^3z}{6} - \frac{53\,x^3yz}{6} + \frac{95\,x^4y}{4} + \frac{7\,x^3y^2}{8} \\
&-\frac{7\,y^3z^2}{24} + \frac{xy^2z^2}{12} + \frac{7\,x^3z^2}{24} - z^5 + \frac{71\,x^4z}{12} \qquad (8.3)
\end{aligned}
$$

The Jacobian matrix of this nonlinear system at the origin is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This matrix possesses a zero eigenvalue and a pair of purely imaginary eigenvalues. Without any simplification, this system looks very complicated. However, if the transformation

$$
\begin{aligned}
\bar{x} \;=\;& x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^5 \qquad (8.4)\\
\bar{y} \;=\;& y - y^2 + \frac{1}{2}y^3 - \frac{1}{6}y^5 \qquad (8.5)\\
\bar{z} \;=\;& z - xy - z^2 + \frac{x^2y}{2} + \frac{xy^2}{2} + \frac{z^3}{2} + xyz - \frac{xy^2z}{3} \\
&-\frac{2\,x^2y^2}{3} - \frac{5\,xyz^2}{6} - \frac{x^2yz}{3} + \frac{xyz^3}{6} - \frac{z^5}{6} + \frac{7\,xy^2z^2}{24} \\
&-\frac{x^4y}{8} + \frac{7\,x^3y^2}{24} - \frac{x^3yz}{12} + \frac{7\,x^2y^3}{24} + \frac{7\,x^2y^2z}{12} \\
&+\frac{7\,x^2yz^2}{24} - \frac{xy^4}{8} - \frac{xy^3z}{12} \qquad (8.6)
\end{aligned}
$$

is applied to (8.1)-(8.3), we obtain a new system as follows:

$$\dot{\bar{x}} = -\bar{y} - \left(\bar{x}^2 + \bar{y}^2\right)(3\,\bar{y} + 4\,\bar{x}) + O(|\cdot|^4) \tag{8.7}$$

$$\dot{\bar{y}} = \bar{x} + \left(\bar{x}^2 + \bar{y}^2\right)(3\,\bar{x} - 4\,\bar{y}) + O(|\cdot|^4) \tag{8.8}$$

$$\dot{\bar{z}} = -\bar{z}^5 + O(|\cdot|^6). \tag{8.9}$$

If we take

$$\bar{V}(\bar{x}, \bar{y}, \bar{z}) = (\bar{x}^2 + \bar{y}^2)^2 + \bar{z}^2 \tag{8.10}$$

as a Lyapunov function candidate for the system (8.7)-(8.9), it is straightforward to verify that

$$\frac{d\bar{V}(\bar{x}, \bar{y}, \bar{z})}{dt} = -16\,(\bar{x}^2 + \bar{y}^2)^3 - 2\,\bar{z}^6 + O(|\cdot|^7). \tag{8.11}$$

The LASY of the system (8.7)-(8.9) (or equivalently, that of (8.1)-(8.3)) follows since (8.11) is locally negative definite. A Lyapunov function in the original coordinates (x, y, z) can be derived by just substituting equations (8.4)-(8.6) into (8.10). ◇◇◇

From this example, we can make the following observations. As before, the critical nonlinear systems considered here are supposed to have following structure:

$$\dot{x}_c = A^c x_c + f_c^2(x_c, x_s) + \cdots \tag{8.12}$$

$$\dot{x}_s = A^s x_s + f_s^2(x_c, x_s) + \cdots \tag{8.13}$$

where all of the eigenvalues of A^s are stable, while all of the eigenvalues of A^c are critical. By choosing an appropriate transformation of the the form:

$$\bar{x}_c = x_c + v_c^2(x_s, x_c) + \cdots + v_c^k(x_s, x_c)$$

$$\bar{x}_s = x_s + v_s^2(x_s, x_c) + \cdots + v_s^k(x_s, x_c)$$

the dynamics for $(\bar{x}_c, \bar{x}_s)$ satisfy

$$\dot{\bar{x}}_c = A^c \bar{x}_c + g_c^2(\bar{x}_c) + \cdots + g_c^k(\bar{x}_c) + O(|\bar{x}_c, \bar{x}_s|^{k+1}) \tag{8.14}$$

$$\dot{\bar{x}}_s = A^s \bar{x}_s + g_s^2(\bar{x}_c, \bar{x}_s) + \cdots + g_s^k(\bar{x}_c, \bar{x}_s) + O(|\bar{x}_c, \bar{x}_s|^{k+1}). \tag{8.15}$$

Theorem 32 *Assume that the subsystem (8.14) is LAS without the necessity of considering the higher order terms, and furthermore a Lyapunov function $\bar{V}(\bar{x}_c)$ exists such that*

$$\dot{\bar{V}}(\bar{x}_c) = \frac{\partial \bar{V}(\bar{x}_c)}{\partial \bar{x}_c}(A^c \bar{x}_c + g_c^2(\bar{x}_c) + \cdots + g_c^k(\bar{x}_c)) \tag{8.16}$$

is a homogeneous polynomial of degree $2r$. Let $P = P^T > 0$ such that $\bar{V}(\bar{x}_s) = \bar{x}_s^T P \bar{x}_s$ is a Lyapunov function for the linear system $\dot{\bar{x}}_s = A^s \bar{x}_s$. Under these hypothesis, the function $\bar{V}(\bar{x}_c, \bar{x}_s) = (\bar{x}_s^T P \bar{x}_s)^r + \bar{V}(\bar{x}_c)$ is a Lyapunov function for the system (8.14)-(8.15). $\diamond\diamond\diamond$

In [23], Fu has also done some research on the construction of Lyapunov functions for critical nonlinear systems with multiple eigenvalues when A^c is diagonalizable. That work is similar to what is presented below in Corollary 1 except for the fact that it can only be applied to critical systems for which the LAS can be determined by the first through third order terms. After those systems are transformed into some sort of normal forms, the construction of Lyapunov functions for these classes of critical nonlinear systems can be easily understand. In fact, the algorithm presented in Chapters 7 for the construction of Lyapunov functions for CASE 1 and CASE 2 critical nonlinear systems is only a special case of Theorem 32.

Corollary 1 *Consider a critical nonlinear system of the form (8.12) and (8.13) with (without of loss of generality)* $A^c = diag(\Lambda_1, \Lambda_2, \cdots, \Lambda_{n_c})$, *where for* $j = 1, 2, \cdots, n_c$, Λ_j *is either a scalar zero or* $\Lambda_j = \begin{pmatrix} 0 & -\omega_j \\ \omega_j & 0 \end{pmatrix}$ *with* $\omega_j \in R^+$. *If* x_{cj} $(j = 1, 2, \cdots, n_c)$ *is the state cooresponding to* Λ_j, *and the critical dynamics* $\bar{x}_{cj}$ *in a (partial) approximate normal form of the form (8.14) and (8.15) satisfy*

$$\dot{\bar{x}}_{cj} = \sum_{i=2}^{2k_j+1} a_i \bar{x}_{cj}^i + O(|\bar{x}|^{2k_j+2}) \tag{8.17}$$

if $\Lambda_j = 0$ *where* $a_{j,2} = \cdots = a_{j,2k_j} = 0$ *but* $a_{j,2k_j+1} < 0$, *or*

$$\dot{\bar{x}}_{cj,1} = -\omega_j \bar{x}_{cj,2} + \sum_{i=1}^{k_j} (m_{j,2i+1} \bar{x}_{cj,1} + n_{j,2i+1} \bar{x}_{cj,2}) + O(|\bar{x}|^{2k_j+2}) \tag{8.18}$$

$$\dot{\bar{x}}_{cj,2} = \omega_j \bar{x}_{cj,1} + \sum_{i=1}^{k_j} (m_{j,2i+1} \bar{x}_{cj,2} - n_{j,2i+1} \bar{x}_{cj,1}) + O(|\bar{x}|^{2k_j+2}) \tag{8.19}$$

if $\Lambda_j = \begin{pmatrix} 0 & -\omega_j \\ \omega_j & 0 \end{pmatrix}$ *where* $m_{j,3} = \cdots = m_{j,2k-1} = 0$ *but* $m_{j,2k_j+1} < 0$, *then the nonlinear system (8.14) and (8.15) (or equivalently (8.12) and (8.13)) is LAS.* ◇◇◇

8.2 Normal Forms For Discrete-Time Nonlinear Systems

In this section, an introduction to normal form techniques for discrete-time nonlinear systems is provided.

Consider a discrete-time system represented as follows,

$$x \mapsto f(x) = Ax + f^2(x) + \cdots + f^k(y) + \cdots. \tag{8.20}$$

Here $\mapsto$ represents iterative transition. Let

$$x = y + P^k(y)$$

be a near identity change of coordinates. The dynamics for y will take the form

$$y \mapsto Ay + g^2(y) + \cdots + f^k(y) - \{P^k(Ay) - AP^k(y)\} + g^{k+1}(y) + \cdots \tag{8.21}$$

where $g^i = f^i$ for $i = 2, 3, \cdots, k-1$, $g^k(y) = f^k(y) - \{P^k(Ay) - AP^k(y)\}$, and $g^i \neq f^i$ in general for $i \geq k+1$. Therefore, the linear operator $L_A^k : H(n,n,k) \to H(n,n,k)$ can be defined by

$$L_A^k P^k(y) = P^k(Ay) - AP^k(y). \tag{8.22}$$

For discrete time systems, a eigenvalue λ is called critical if and only if its magnitude $\|\lambda\| = 1$.

Definition 8 [4] Let $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the spectrum of the $n \times n$ matrix A. Let $\alpha = [\alpha_1, \cdots, \alpha_n]$ be a set of nonnegative integers. A monomial $x^\alpha e_j$, with $\sum_{i=1}^n \alpha_i = k \geq 2$ is called a resonant monomial if

$$\lambda^\alpha \stackrel{def}{=} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} = \lambda_j \tag{8.23}$$

for some integers k and j. This condition is called the resonant condition for the matrix A in discrete-time domain. ◊◊◊

One of the fundamental theorems about normal forms for discrete-time systems is given as follows.

Theorem 33 [4] If A is an upper triangular with (ordered) diagonal elements $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *then a normal form map can be chosen such that its nonlinear part contains only resonant monomials.* ◇◇◇

Without any difficulty, one can prove the following theorem.

Theorem 34 Consider a discrete-time critical nonlinear system:

$$\begin{pmatrix} x_c \\ x_s \end{pmatrix} \mapsto \begin{pmatrix} A^c x_c + f_c^2(x_c, x_s) + f_c^3(x_c, x_s) + \cdots \\ A^s x_s + f_s^2(x_c, x_s) + f_s^3(x_c, x_s) + \cdots \end{pmatrix} \tag{8.24}$$

where all of the eigenvalues of A^c *are critical while all of the eigenvalues of* A^s *are stable. Any normal form for (8.24) up to degree k has the following structure:*

$$\begin{pmatrix} \bar{x}_c \\ \bar{x}_s \end{pmatrix} \mapsto \begin{pmatrix} A^c \bar{x}_c + \bar{f}_c^2(\bar{x}_c) + \cdots + + \bar{f}_c^k(\bar{x}_c) + O(\bar{x}_c, \bar{x}_s|^{k+1}), \\ A^s \bar{x}_s + \bar{f}_s^2(\bar{x}_c, \bar{x}_s) + \cdots + + \bar{f}_s^k(\bar{x}_c, \bar{x}_s) + O(\bar{x}_c, \bar{x}_s|^{k+1}) \end{pmatrix}, \tag{8.25}$$

where $\bar{f}_s^i(\bar{x}_c, \bar{x}_s = 0) = 0$ *for* $i = 2, 3, \cdots, k$. ◇◇◇

8.3 Summary

In this chapter we first discussed how to construct Lyapunov functions for LAS nonlinear systems with multiple critical eigenvalues. Following this, an introduction to the normal form techniques for discrete-time critical nonlinear systems was presented.

CHAPTER 9
SUMMARY OF THE DISSERTATION

Nonlinear dynamical models arise from both theoretical research and real applications, including robotics, aircraft control, adaptive control, chemical processes, numerical algorithms, and power systems. The most important analysis for nonlinear systems is the determination of stability. As far as stability is concerned, only critical nonlinear systems need further investigation. For critical nonlinear systems, stability analysis is, up to now, predominantly performed using center manifold and bifurcation theories in combination with Lyapunov techniques. When a dynamical system is asymptotically stable, a Lyapunov function must exist. Many system properties, such as stability region, convergence rate, and robustness, are associated with and can be estimated using Lyapunov functions. The search on the determination of stability properties of critical nonlinear systems or the construction of the associated Lyapunov functions is far from complete.

In this dissertation, critical nonlinear systems are studied using normal form theory. Once a critical nonlinear system is in a normal form, the dynamics of the critical states is decoupled locally from those of the stable states. Similar to results derived using center manifold theory, the stability of the entire nonlinear system is determined by that of the critical dynamics. Thus the normal form method is also a model reduction technique. The relationship between two model reduction techniques, i.e., normal form and center manifold methods, is explored. Compared to the center manifold method, the normal form method provides not only a model reduction technique, but also a direction for the construction

of Lyapunov functions. If the critical dynamics is locally asymptotically stable, and a Lyapunov function is available, then Lyapunov functions for the entire system can be easily constructed as a sum of two indivadule Lyapunov functions corresponding to the critical and stable dynamics, respectively.

The convergence of normal forms for a given nonlinear system is not always guaranteed. Construction of Lyapunov functions for purely critical nonlinear systems is only known in a few special cases. CASE 1 and CASE 2 critical nonlinear systems are studied intensively using approximate normal forms. The explicit stability criteria as well as explicit algorithms for construction of Lyapunov functions are proposed once such a system is put into an appropriate (approximate) normal form. However, it is often desirable to have Lyapunov functions for the original systems. By further studying the normal form transformations, it is shown that Lyapunov functions in the original coordinates can be derived by just substituting an appropriate transformation.

When compared with Lyapunov functions constructed by other researchers for some very simple critical nonlinear systems, the Lyapunov functions we constructed are quite complicated in some sense. Methods for simplification of Lyapunov functions are also developed in this dissertation. Using the results in Chapter 6, the Lyapunov function we derived can be as simple as those constructed by other researchers.

The derivation of normal forms and the associated transformations is very computationally intensive. The issue of reducing the computational complexity is studied. In Chapter 7, an alternative algorithm for the construction of Lyapunov functions is proposed wherein only partial (approximate) normal forms are necessary. The simplification of the Lyapunov functions is also addressed in Chapter 7. A more general of class of critical nonlinear systems is examined in Chapter 8, and some sufficient conditions for determining the local

stability and for constructing Lyapunov functions are obtained. Finally, the use of normal form techniques for discrete-time nonlinear systems analysis is approached in Chapter 8. All of the results obtained in this dissertation for continuous-time critical nonlinear systems will carry over to discrete-time with some modifications.

In conclusion, the following has be accomplished.

- Equivalence of center manifolds.
- Decoupled-like structure of normal forms for critical nonlinear systems.
- Relationship between center manifold and normal form theories.
- Stability analysis of critical nonlinear systems using normal forms.
- A general direction for the construction of Lyapunov functions for critical nonlinear systems
- Two different explicit algorithms for the construction of Lyapunov functions for critical nonlinear systems in two special cases.
- Simplification of Lyapunov functions.
- Simplification of the computation for the construction of Lyapunov functions. normal form techniques.
- Introduction to the use of normal form techniques for discrete-time critical nonlinear systems.

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BIOGRAPHICAL SKETCH

Aiguo Yan was born in Hubei Province, China on October 20th, 1963. He received his Bachelor and Master of Science degrees in electrical and computer engineering from China Textile University, Shanghai, China, in 1983 and 1986, respectively. After graduation, he served as a Systems Engineer in the Institute of Automatic Control, Beijing from January 1986 to December 1992, and a Senior Systems Engineer in Unimac (a joint venture with Unisys USA), Beijing from January 1992 to July 1992. He came to America in August 1992, entering the Ph.D program in the Department of Electrical Engineering and Computer Science at the University of Vermont, Burlington, Vermont. Since August 1993, he has been with the department of Electrical and Computer Engineering at the University of Florida, Gainesville, Florida, pursuing his doctoral degree. His current research interests include stability and stabilization of nonlinear systems, adaptive and decouple control, control system performance analysis, and scientific computing.

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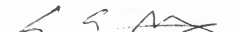

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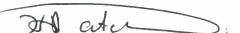

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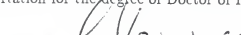

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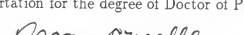

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